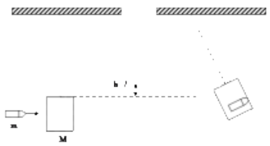
Week 10 (Nov 28 - Dec 2)

Friday, September 23, 2016 2:59 PM

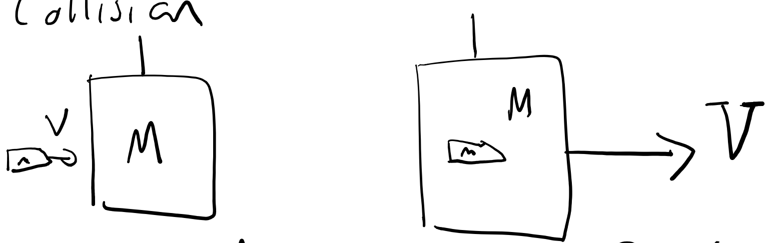
Phys 101- Instructor: M. Özgür Oktel – 2016 QUIZ-19

A bullet of mass m hits a sandbag of mass M that is suspended from the ceiling by a massless string. At the moment the bullet hits it has a horizontal velocity v, and it gets stuck in the sandbag. Find the maximum height the sandbag will rise to during its swing. (Assume that the bullet comes to a stop immediately, Acceleration due to gravity is g=10 m/s^2, m=10 g, M= 1 kg, v=100 m/s.)



Ballistic pendulum

1°) Collision



$P_i = mv$ Momentum is conserved. $P_f = (m+M) V$

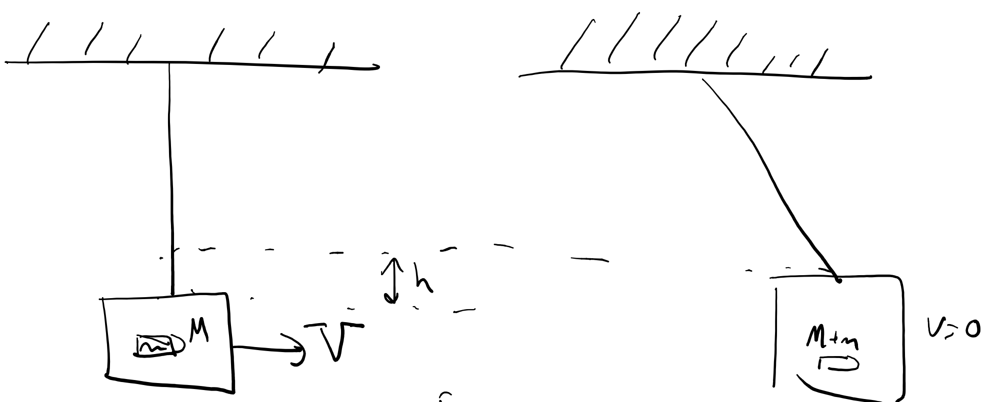
$\frac{1}{2} m v^2$

$V = \frac{m}{(m+M)} v$

$\frac{1}{2}(m+M) V^2$

$= \frac{1}{2} \cancel{(m+M)} \frac{m^2}{(m+M)^{\cancel{2}}} v^2$

2°) Pendulum part



Energy is conserved

$$\frac{1}{2} \cancel{(m+M)} V^2 = \cancel{(m+M)} g h$$

$$h = \frac{V^2}{2g} = \boxed{\frac{1}{2g} \frac{m^2}{(m+M)^2} v^2}$$

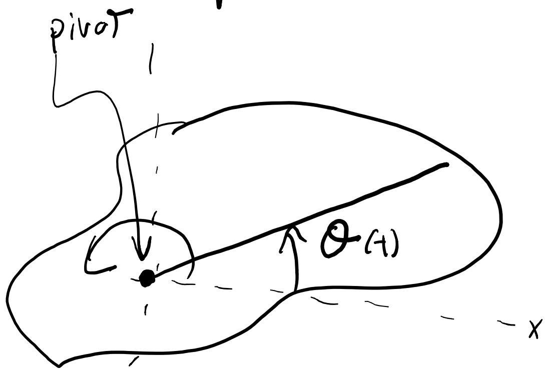
1°) Goal ✓
2°) Units ✓
3°) Limits ✓

$$= \frac{1}{m/s^2}\left(\frac{m}{s}\right)^2 = m \checkmark$$

Rotational Motion

[Thursday 8:40 Lecture]

Rotational quantities



$\theta(t) \rightarrow$ angle, orientation, position, angular position.

$$\theta(t) = \omega t \quad \text{(Uniform circular motion)}$$

Now $\theta(t)$ will be any function!

Angular velocity $\boxed{\omega(t) = \frac{d}{dt}\theta(t)}$

Angular acceleration $\boxed{\alpha(t) = \frac{d}{dt}(\omega(t)) = \frac{d^2}{dt^2}\theta(t)}$

Rotational Motion		Linear Motion
$\alpha(t)$	$\longleftrightarrow$	$a(t)$
~~W~~ $\omega(t)$	$\longleftrightarrow$	$v(t)$

$\cancel{W}$ $\quad \omega(t) \longleftrightarrow v(t)$

omega

$\theta(t) \longleftrightarrow x(t)$

All kinematic formulas can be adopted!

$$\theta(t) = \int_0^t \omega(t')\,dt' + \theta(t=0)$$

$\vdots$

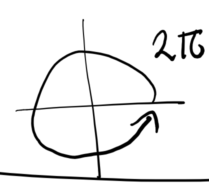
Constant angular acceleration $\alpha(t) = \alpha$

$$\theta(t) = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0$$

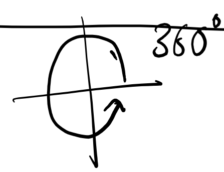
$$x(t) = \frac{1}{2} a t^2 + v_0 t + x_0$$

So kinematics is simple!

$[\theta] \longrightarrow$ $\boxed{\text{radians (convention)}}$



degree



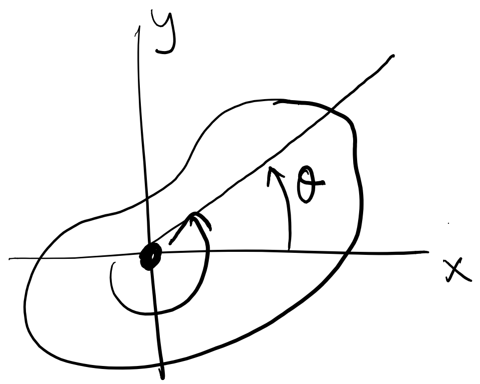
unitless

$[\omega] = 1/\text{sec} = Hz$

~~Billy Radio 96.6 MHz 102 10^6 Hz~~

$[\alpha] = 1/\text{sec}^2$

Are these quantities scalars or vectors?

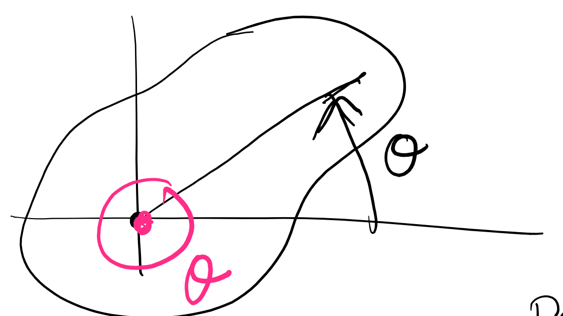


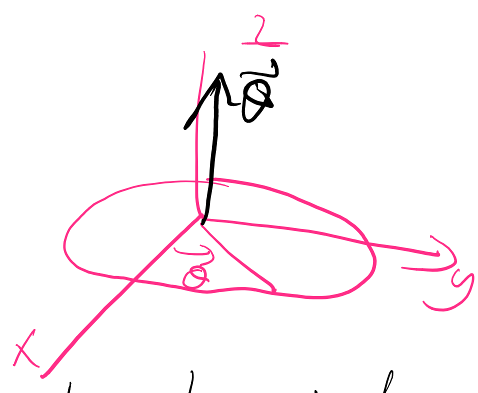
you can regard $\theta(t)$ as a scalar if one is rotating around a <u>fixed axis</u>

same for $\omega(t)$, $\alpha(t)$

More complicated (More general case)

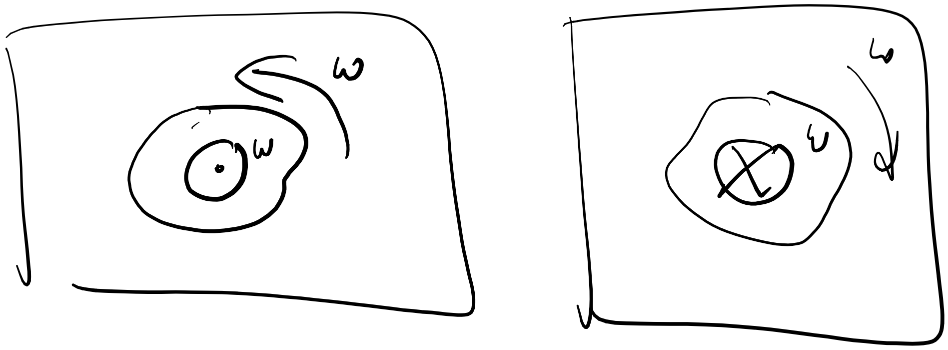
$\vec{\theta}$, $\vec{\omega}$, $\vec{\alpha}$





points along the axis of rotation.

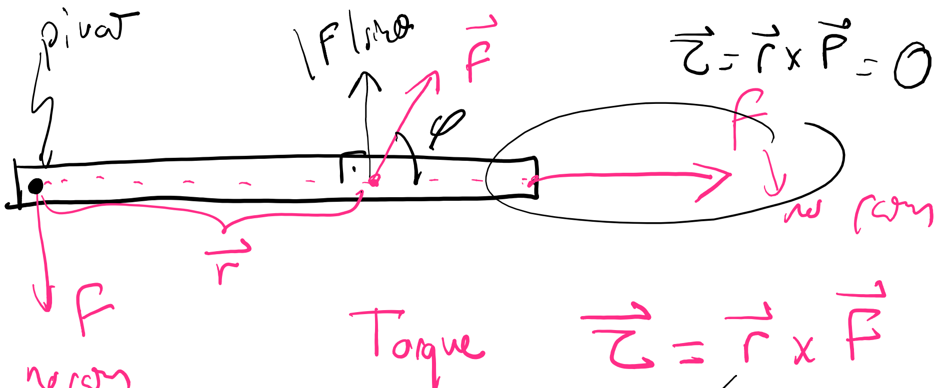
(PHYS 101 we can mostly use the scalar notation!)



$\theta,\ \omega,\ \alpha$ are kinematic variables.

$\vec{F} = m\vec{a} \implies$ Rotational? $\vec{a} \to \alpha$

Rotational equivalent of <u>Force</u>



Torque $\vec{\tau} = \vec{r} \times \vec{F}$

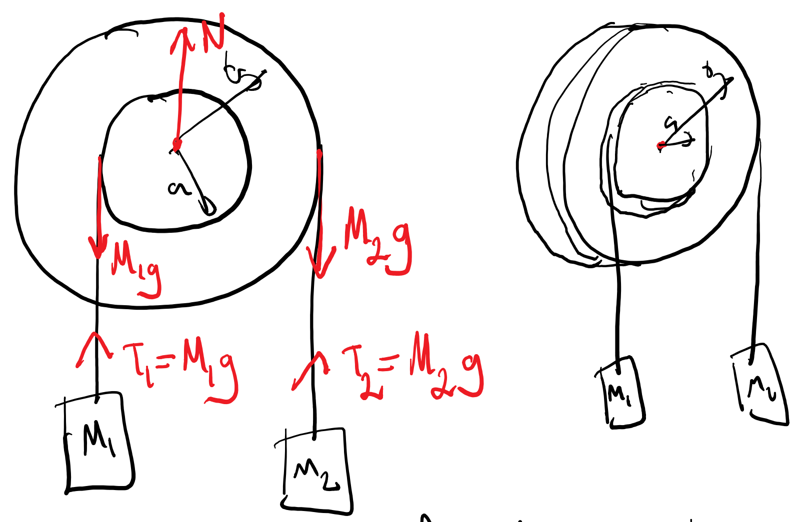
$\vec{r}$: Vector from the pivot to the point where force is applied

$\vec{F}$: Applied force.

$|\vec{\tau}| = |\vec{r}|\underbrace{|\vec{F}|\sin(\varphi)}_{\text{Measure of the perpendicular component!}}$

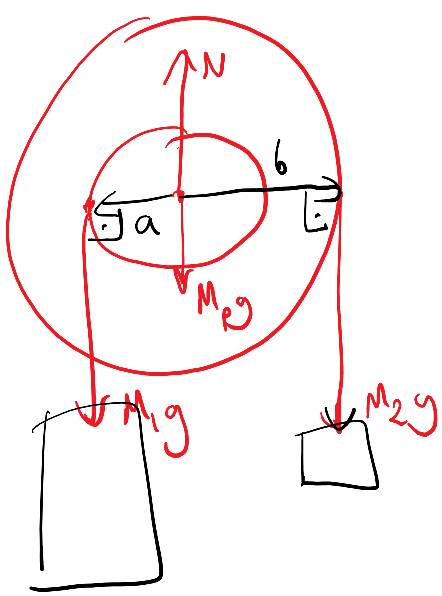
Direction can be used to determine sense of rotation.

Ex



What should be the ratio of $\frac{M_1}{M_2}$ so that the system is in equilibrium?

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$
$$= -\vec{B} \times \vec{A}$$



$$\sum \tau = 0$$

$$\tau_2 = \vec{b} \times M_2 \vec{g}$$

$$\tau_1 = \vec{a} \times M_1 \vec{g}$$

$$|\vec{\tau}_1| = |\vec{\tau}_2|$$

$$b M_2 \cancel{g} = a M_1 \cancel{g}$$

$$\boxed{\frac{M_1}{M_2} = \frac{b}{a}}$$

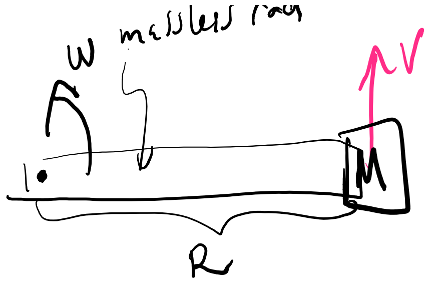
What is the equivalent of mass for rotation?

massless rod

w

$\uparrow v$

What is the kinetic energy of this



What is the kinetic energy of this system.

$v = \omega R$

$$K = \frac{1}{2} M v^2$$

$$K = \frac{1}{2} M (R^2 \omega^2)$$

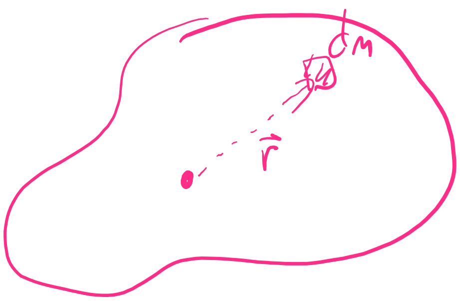
$$= \frac{1}{2} \underbrace{(MR^2)}_{I = MR^2} \omega^2$$

$$\boxed{I = MR^2}$$

K $\qquad$ K_{rot}

$\frac{1}{2} m v^2$ $\qquad$ $K_{rot} = \frac{1}{2} I \omega^2$

Moment of inertia!



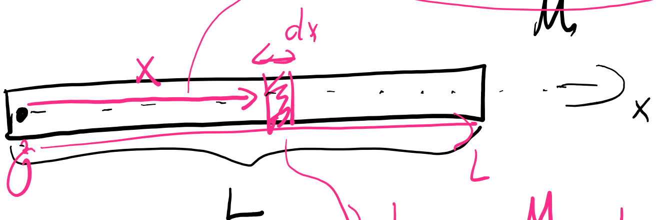
$$\boxed{I = \int r^2 \, dm}$$

$$\vec{F} = m\vec{a} \Longleftrightarrow \boxed{\vec{\tau} = I\vec{\alpha}}$$

Moment of inertia.

$$I = \int r^2 \, dm$$

Find the moment of inertia for a rod, uniform density through its edge.



$$I = \int r^2 \, dm$$

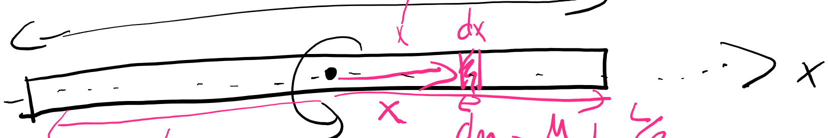
$$dm = \frac{M}{L} dx$$

$$I_{edge} = \int_0^L x^2 \frac{M}{L} dx = \frac{M}{L} \int_0^L x^2 dx = \frac{M}{L} \frac{x^3}{3} \Big|_0^L$$

$$= \frac{M L^3}{L \, 3} = \boxed{\frac{1}{3} M L^2}$$

* Find I for the same rod rotating around its center of mass

$$I = \int r^2 \, dm$$



$$dm = \frac{M}{L} dx$$

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \frac{M}{L} dx = \frac{M}{L} \frac{x^3}{3} \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{M}{3L} \left[\left(\frac{L}{2}\right)^3 - \left(\frac{-L}{2}\right)^3 \right]$$

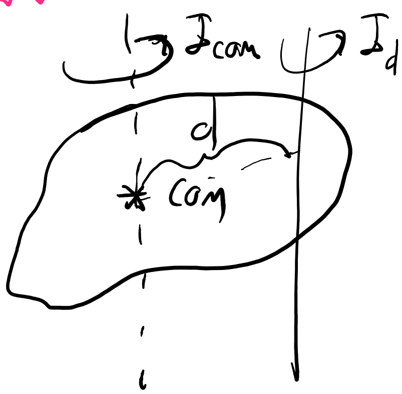
$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \frac{M}{L}\, dx = \frac{M}{L} \frac{x^3}{3}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(\frac{-L}{2}\right)^3\right]$$

$$= \frac{M}{3L}\frac{2L^3}{8}$$

$$\boxed{I_{com} = \frac{ML^3}{12}}$$

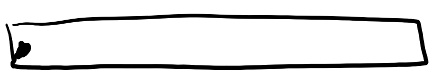
$$\boxed{I_{edge} \neq I_{com}}$$

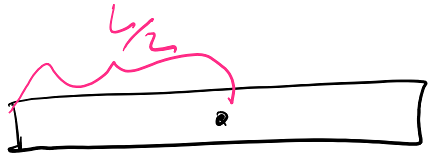
Parallel axis theorem



$$\boxed{I_d = I_{com} + Md^2}$$

I_{com} is the smallest possible I





$I_{edge} = \frac{1}{3} ML^2$

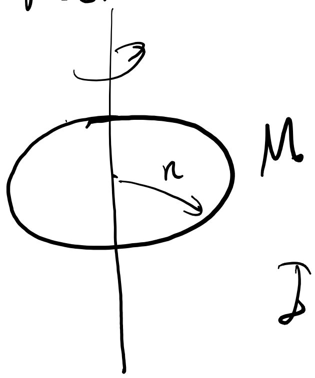
$I_{com} = \frac{1}{12} ML^2$

$$I_{edge} = I_{com} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{1}{12} ML^2 + M\frac{L^3}{4}$$

$$= \left(\frac{1}{12}+\frac{3}{12}\right)Ml^2 = \frac{\overset{4}{1}}{3}Ml^2$$

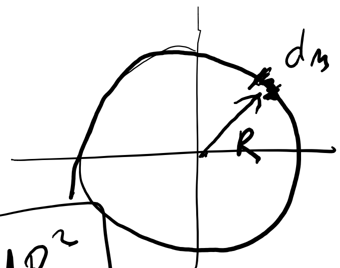
Moment of inertia for a ring



M

$$I = ?$$

top view



$$I = \int r^2\, dm$$

$$= R^2 \underbrace{\int dm}_{M} = \boxed{MR^2}$$

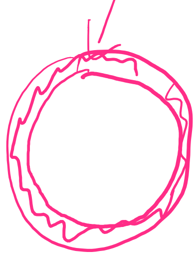
Moment of inertia for a uniform disc!

Top

$$I = \int r^2\, dm$$

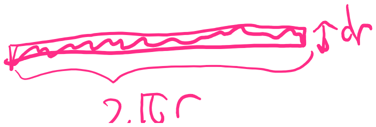
$$dI = (dm)\, r^2$$

$$dm = \frac{M}{\pi R^2}\, dA = \frac{M}{\pi R^2}\, 2\pi r\, dr$$



$$dA = \pi (r+dr)^2 - \pi r^2$$

$$= \cancel{\pi r^2} + \pi 2r\, dr + \cancel{\pi (dr)^2} - \cancel{\pi r^2}$$



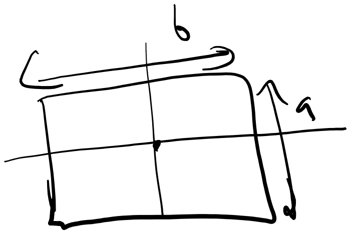
$$= 2\pi r\, dr$$

$$dI = r^2\,dm = r^2 \frac{M}{\pi R^2} 2\pi r\,dr$$

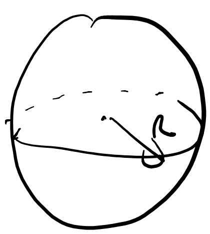
$$I = \int_0^R r^2 \frac{M}{\cancel{\pi} R^2} 2\cancel{\pi} r\,dr = \frac{2M}{R^2} \underbrace{\int_0^R r^3\,dr}_{\left.\frac{r^4}{4}\right|_0^R}$$

$$= \frac{2M}{R^2}\,\frac{R^4}{4}$$

$$\boxed{I_{com} = \frac{MR^2}{2}}$$



$$I_{rect} = \frac{1}{12} M\left(a^2+b^2\right)$$



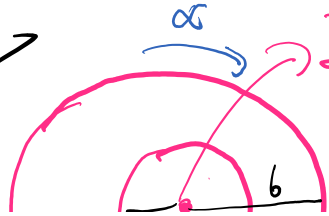
$$I_{sphere} = \frac{2}{5} MR^2$$

$$\boxed{\vec{\tau} = I\,\vec{\alpha}}$$

Ex

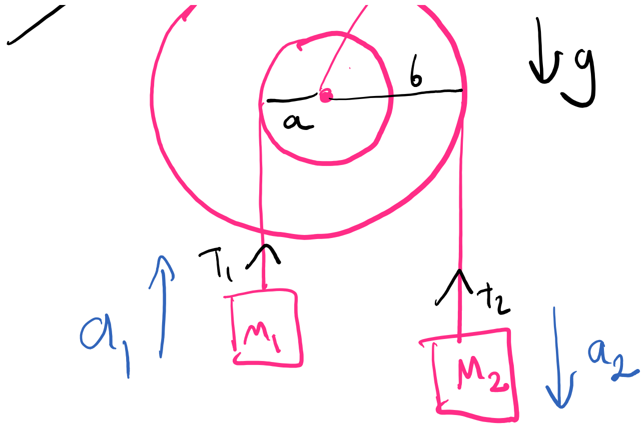
α

$I \Longrightarrow$ moment of inertia of the wheel.



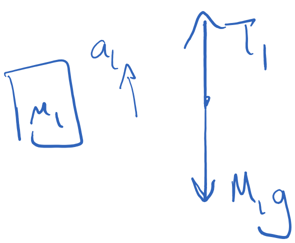
$\downarrow g$

Find T_1, T_2

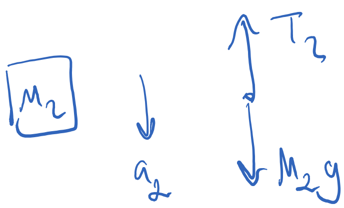


Find T_1, T_2
and angular acceleration
of the wheel!

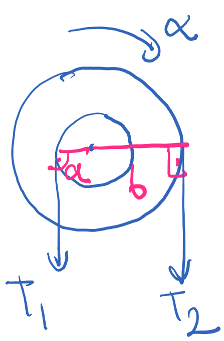
Free body diagrams



$$\boxed{M_1 a_1 = T_1 - M_1 g}$$ (1)



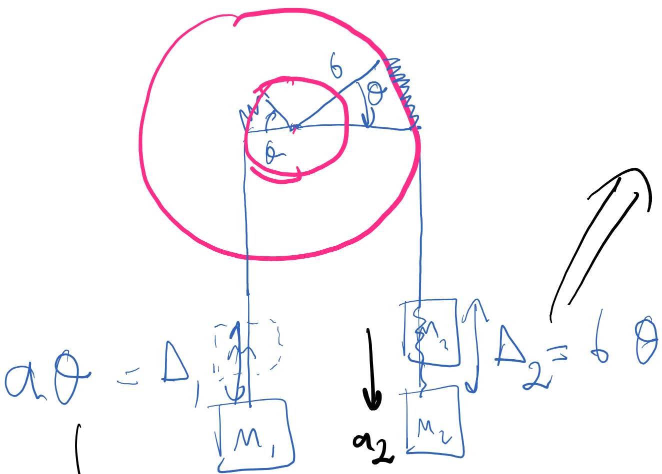
$$\boxed{M_2 a_2 = M_2 g - T_2}$$ (2)



$$\tau = I\alpha$$

$$\boxed{T_2 b - T_1 a = I\alpha}$$ (3)

a_1, a_2 and α must be related!



$$\frac{d^2}{dt^2}(\Delta_2) = a_2 = b\underbrace{\frac{d^2\theta}{dt^2}}_{\alpha}$$

$$\boxed{a_2 = b\alpha}$$

$$\boxed{a_1 = a\alpha}$$

$$M_1 a_1 = T_1 - M_1 g \xrightarrow{a} (M_1 a) a\alpha = T_1 a - M_1 a g$$

$$M_2 a_2 = M_2 g - T_2 \xrightarrow{b} (M_2 b) b\alpha = M_2 b g - T_2 b$$

$$I\alpha = T_2 b - T_1 a$$

$$(M_1 a^2)\alpha = \cancel{T_1 a} - M_1 a g$$

$$(M_2 b^2)\alpha = M_2 b g - \cancel{T_2 b}$$

$$+ \quad I\alpha = \cancel{T_2 b} - \cancel{T_1 a}$$

$$(I + M_1 a^2 + M_2 b^2)\alpha = (M_2 b - M_1 a) g$$

$$\boxed{\alpha = \frac{(M_2 b - M_1 a)}{(I + M_1 a^2 + M_2 b^2)} g}$$

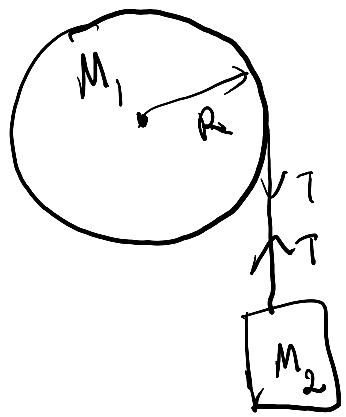
If $M_1 a = M_2 b$ ✓

$\alpha = 0$

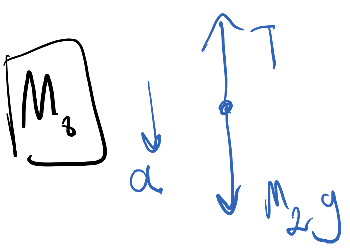
$$a_1 = a\,\alpha \qquad a_2 = b\,\alpha$$

$$T_1 = M_1 g + M_1 a_1 \qquad T_2 = M_2 g - M_2 a_2$$

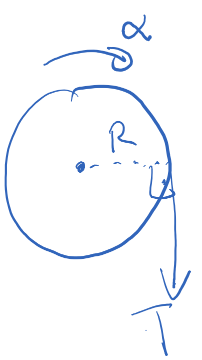
Ex



Find the acceleration of the mass M_2.



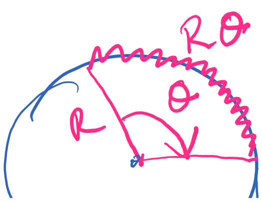
$$\boxed{M_2 g - T = a}$$



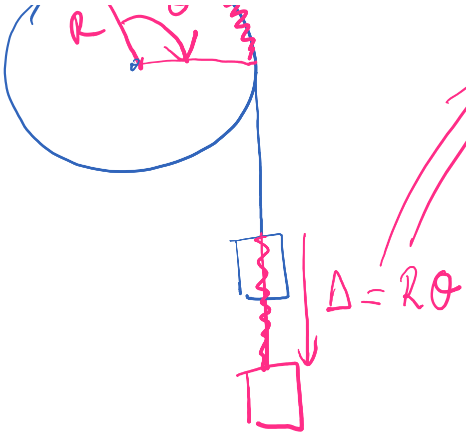
$$\tau = TR = I\,\alpha \qquad I = \frac{M_1 R^2}{2}$$

$$T\cancel{R} = \frac{M_1 R^{\cancel{2}}}{2}\alpha$$

$$\boxed{\frac{M_1 R}{2}\alpha = T}$$



$$\frac{d}{dt}(\Delta) = R\,\frac{d}{dt}(\theta)$$



$$\underbrace{\frac{d}{dt}(\Delta)}_{} = R\frac{d}{dt}(\theta)$$

$$v = R\omega$$

$$\frac{d}{dt}v = R\frac{d\omega}{dt}$$

$$\boxed{a = R\alpha}$$

$$\alpha = \frac{a}{R}$$

$$\boxed{M_2 g - T = M_2 a} \quad (1)$$

$$\boxed{T = \frac{M_1 R}{2}\alpha} \Longrightarrow \boxed{T = \frac{M_1}{2} a} \quad (2)$$

$$M_2 g - \frac{M_1}{2} a = M_2 a$$

$$M_2 g = \left(M_2 + \frac{M_1}{2}\right) a$$

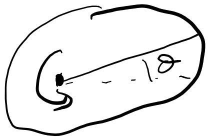
$$\boxed{a = \frac{M_2}{\left(M_2 + \frac{M_1}{2}\right)} g} \qquad \boxed{T = \frac{M_1 M_2}{(2M_2 + M_1)} g}$$

1°) Gad ✓
2°) Units ✓
3°) Limits ✓

$M_1 \to 0 \quad a = g$ ✓

Rotational Motion

$\theta(t)$; $X(t)$



$\theta(t)$; $x(t)$

$\frac{d\theta}{dt} = \omega(t) \rightarrow$ angular velocity ; $v(t)$

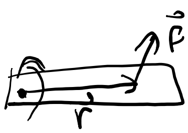
$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \alpha(t) \rightarrow$ angular acceleration. ; $a(t)$

angular acc.

Dynamics: $\vec{\tau} = I\vec{\alpha} \Leftrightarrow \vec{F} = m\vec{a}$

torque

$\vec{\tau} = \vec{r} \times \vec{F}$



Moment of inertia

$$I = \int r^2 dm$$

$I = $ $I_{ring} = MR^2$; $I_{Disc} = \frac{MR^2}{2}$; $I_{rod} = \frac{ML^2}{12}$

parallel axis theorem

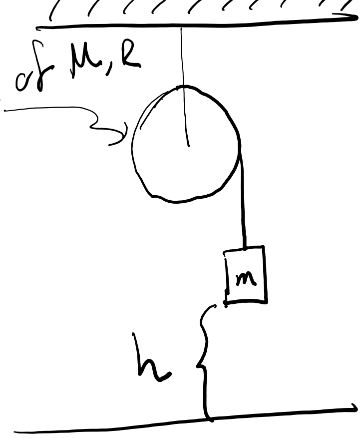
$$I_d = I_{com} + Md^2$$

Kinetic Energy of Rotation

$$K = \frac{1}{2} I \omega^2$$

[But we have to be careful about which I and which ω!]

EX



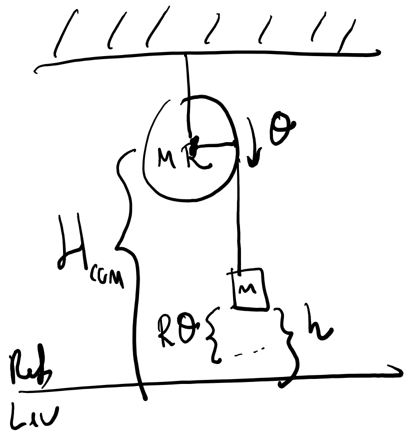
How fast does m hit the floor if it starts from the rest configuration shown?

Potential energy the

Solution 1

Energy is conserved

Potential energy depends on the position of COM!

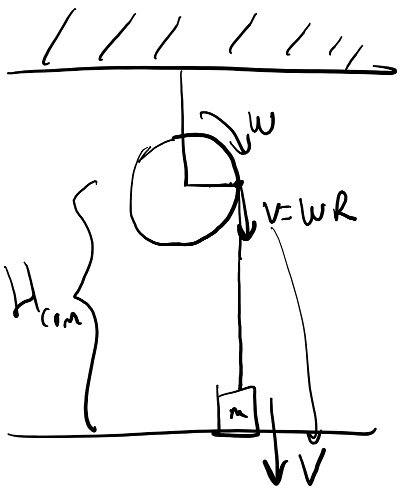


$$E_i = mgh + MgH_{com}$$

$$\frac{d\theta}{dt} = \omega \qquad \frac{d}{dt}(R\theta) = v$$

$$\boxed{R\omega = v}$$

$$I_{disc} = \frac{MR^2}{2}$$



$$E_f = MgH_{com} + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= MgH_{com} + \frac{1}{2}mv^2 + \frac{1}{4}MR^2\omega^2$$

(1) $E_i = E_f$

(2) $\omega R = v$

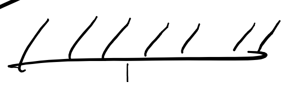
$\alpha R = a$

$$\cancel{MgH_{com}} + mgh = \cancel{MgH_{com}} + \frac{1}{2}mv^2 + \frac{1}{4}MR^2\omega^2$$

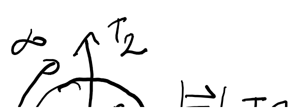
$$mgh = \frac{1}{2}mv^2 + \frac{1}{4}Mv^2$$

$$mgh = \frac{1}{2}\left(m+\frac{M}{2}\right)v^2 \Rightarrow \boxed{v = \sqrt{\frac{2mgh}{\left(m+\frac{M}{2}\right)}}}$$

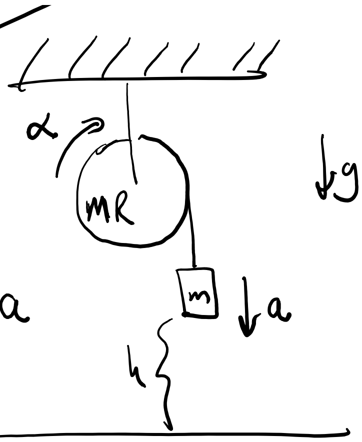
Solution 2

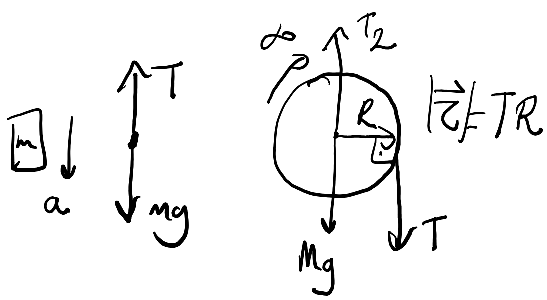


$\uparrow T$



Sol:





$\alpha R = a$

$ma = mg - T$

$I\alpha = TR$

$\frac{MR^{\cancel{2}}}{2}\alpha = T\cancel{R}$

$\frac{Ma}{2} = T$

$$ma = mg - \frac{M}{2}a$$

$$\boxed{a = \frac{m}{\left(m + \frac{M}{2}\right)} g}$$

How long will it take for m to hit the floor?

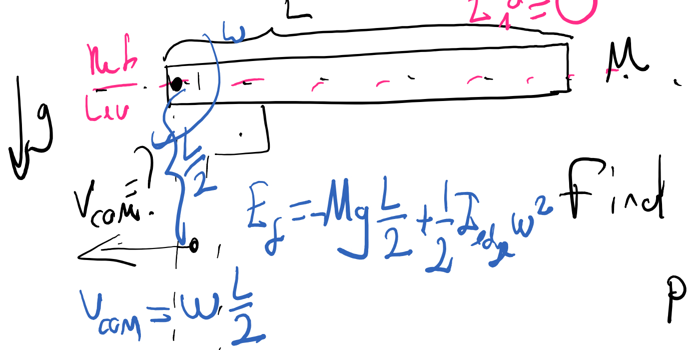
$$\frac{1}{2} a t_f^2 = h$$

$$\boxed{t_f = \sqrt{\frac{2h}{a}}}$$

$$V_f = a t_f = \sqrt{2ha}$$

$$\boxed{V = \sqrt{\frac{2mhg}{\left(m + \frac{M}{2}\right)}}}$$

$E_i = 0$



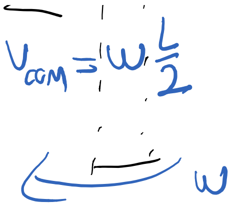
uniform rod

$E_f = -Mg\frac{L}{2} + \frac{1}{2} I_{edge}\omega^2$

Find V_{com} as the rod passes through vertical

$V_{com} = \omega \frac{L}{2}$

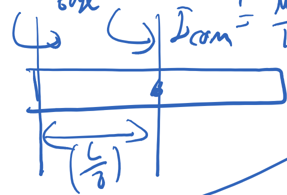
$\bar{v}_{com} = \omega \frac{L}{2}$



passes through vertical configuration.

$I_{edge} = I_{com} + M\frac{L^2}{4} = M\frac{L^2}{3}$

$I_{com} = \frac{ML^2}{12}$



Solution Energy conservation.

$$E_i = E_f$$

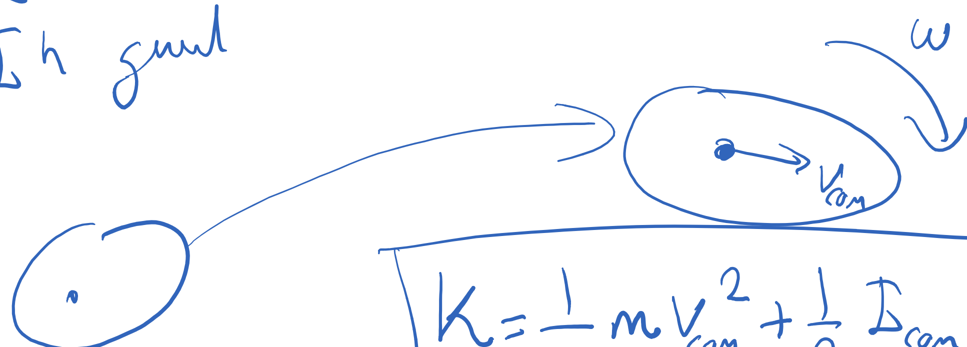
$$0 = -Mg\frac{L}{2} + \frac{1}{2} I_{edge}\,\omega^2$$

$$\cancel{M} g \frac{\cancel{L}}{\cancel{2}} = \frac{1}{\cancel{2}} \frac{\cancel{M}L^{\cancel{2}}}{3}\omega^2$$

$$\sqrt{\frac{3g}{L}} = \omega \implies v_{com} = \omega \frac{L}{2}$$

$$= \sqrt{\frac{3gL^{\cancel{2}}}{\cancel{L}4}} = \boxed{\sqrt{\frac{3}{4}}\sqrt{gL}}$$

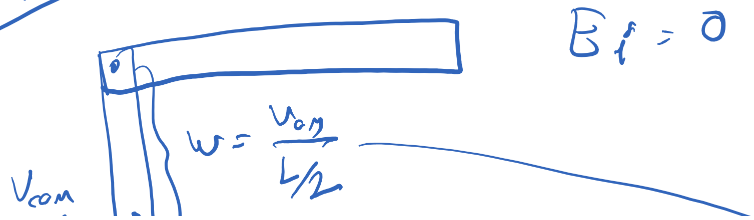
In general



$$\boxed{K = \frac{1}{2} m v_{com}^2 + \frac{1}{2} I_{com}\,\omega^2}$$

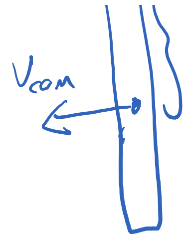
only works for COM.

Solution



$E_i = 0$

$\omega = \frac{v_{com}}{L/2}$



$\omega = \frac{v_{com}}{L/2}$

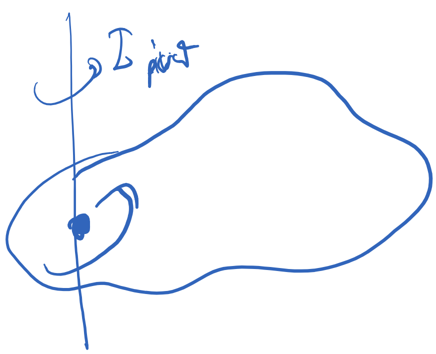
$$E_i = E_f = -Mg\frac{L}{2} + \frac{1}{2} M v_{com}^2 + \frac{1}{2} I_{com}\,\omega^2$$

$$\cancel{M} g \frac{L}{\cancel{2}} = \frac{1}{\cancel{2}} \cancel{M} v_{com}^2 + \frac{1}{\cancel{2}} \frac{\cancel{M} \cancel{L^2}}{12} \frac{v_{com}^2}{\cancel{L^2}/4}$$

$$gL = v_{com}^2\left(1+\frac{1}{3}\right) \Rightarrow v_{com} = \sqrt{\frac{3}{4} g L}$$

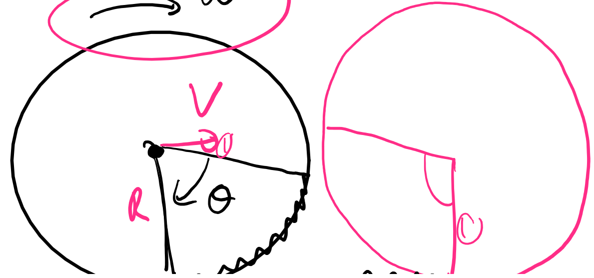
$$\boxed{K = \frac{1}{2} m v_{com}^2 + \frac{1}{2} I_{com}\,\omega^2}$$

Always correct

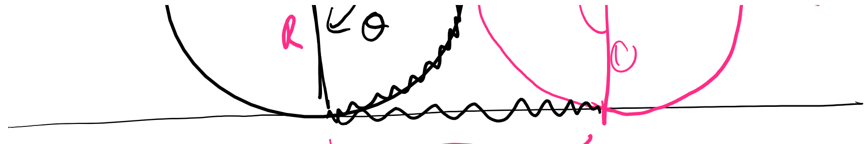


$$K = \frac{1}{2} I_{pivot}\,\omega^2$$

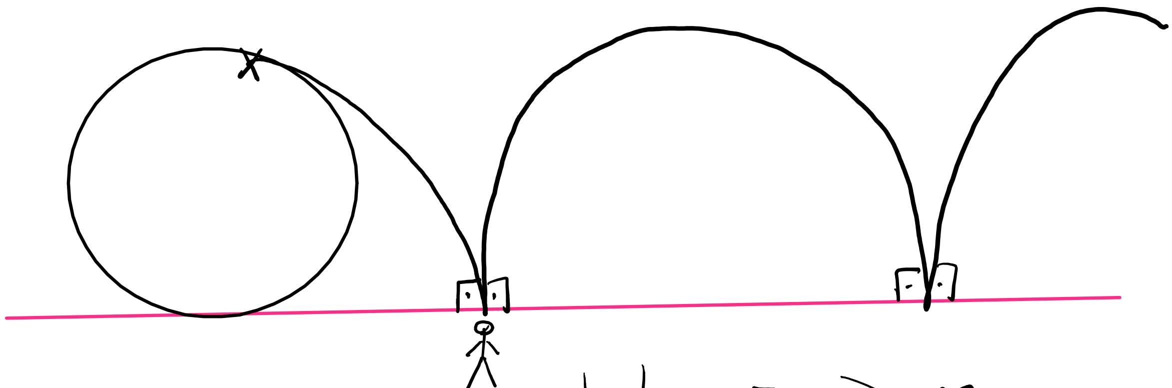
Rolling



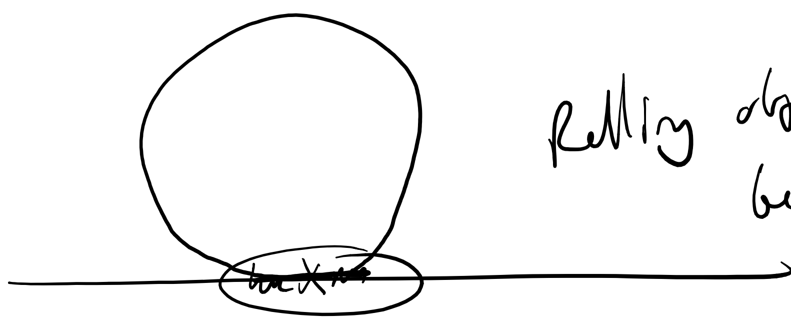
$$\boxed{V = \omega R}$$



During rolling there may be friction but it does not dissipate energy!



no displacement ⇒ no dissipated energy.

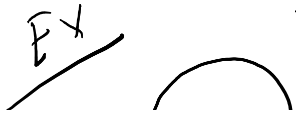


Finite contact

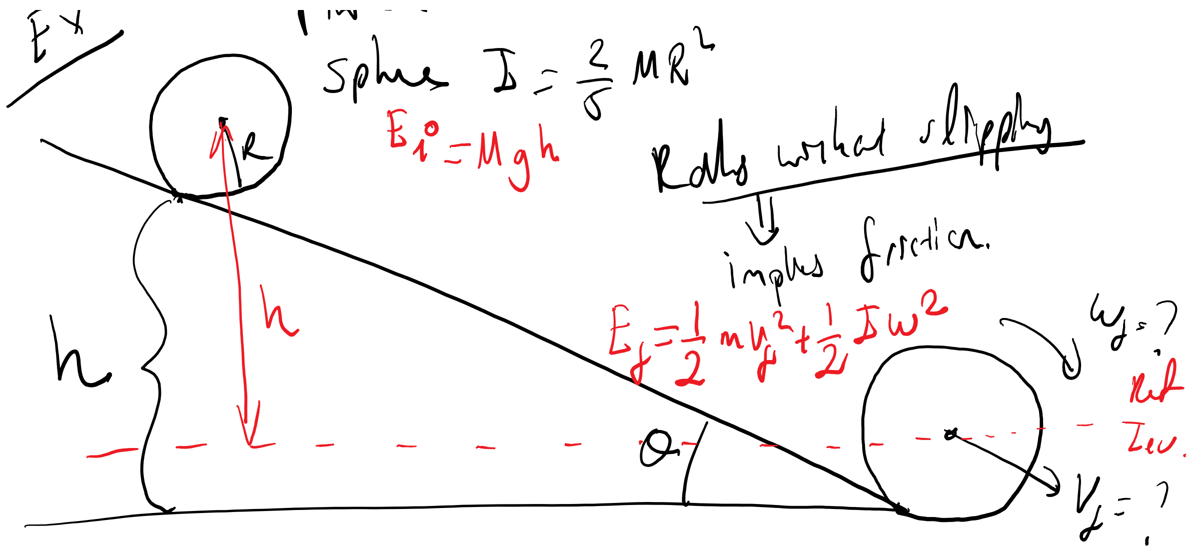
Rolling objects stop because they are not perfect circles.

Ex

Sphere $I = \frac{2}{5} MR^2$



Ex



a) $v_f = ?$ $\quad \omega_f = ?$

b) Find the friction force acting on the sphere during motion.

Although there is friction, during rolling without slipping there is no energy loss, friction force is static friction.

$\Longrightarrow$ Rolling w/o slipping $\Rightarrow$ energy is conserved.

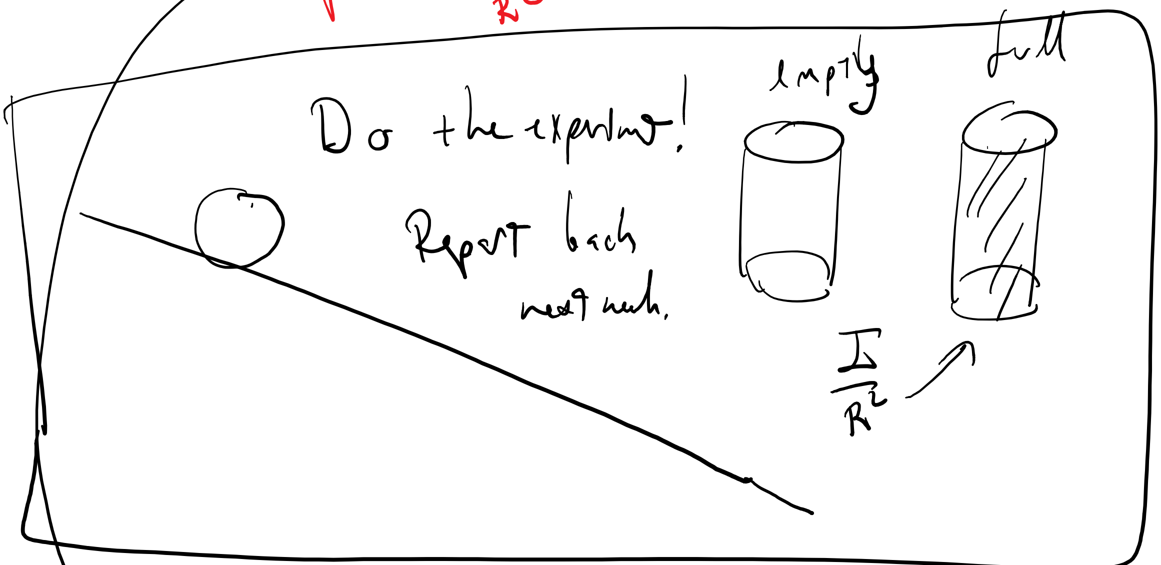
$$Mgh = \frac{1}{2} M v_f^2 + \frac{1}{2} I \omega_f^2$$

$v_f = R\omega_f$

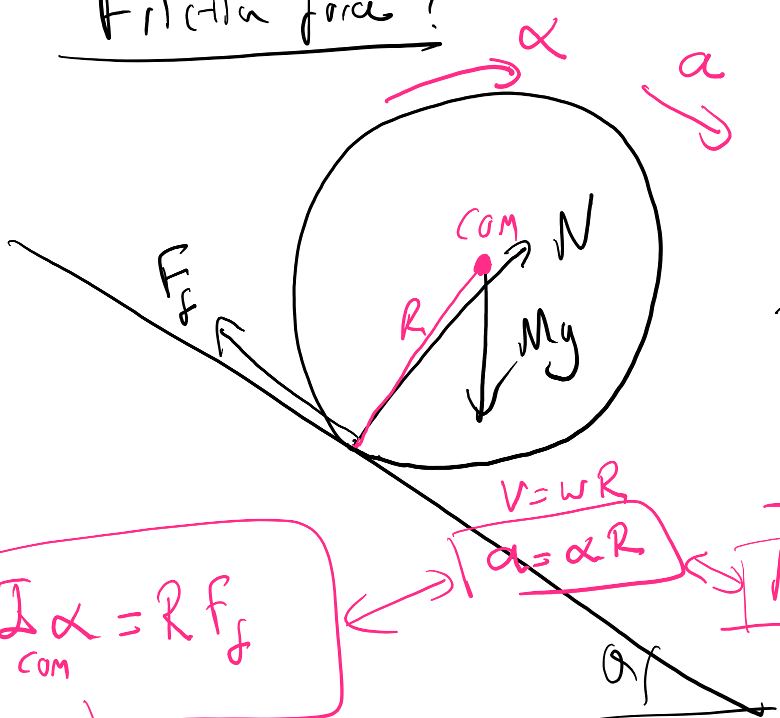
$I = \frac{2}{5} MR^2$

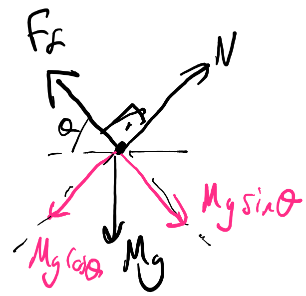
$$Mgh = \frac{1}{2} M v_f^2 + \frac{1}{2} \left(\frac{I}{R^2}\right) v_f^2$$

$$V_f = \sqrt{\frac{2Mgh}{M + \frac{I}{R^2}}} \qquad \omega_f = \sqrt{\frac{2Mgh}{MR^2 + I}}$$



Friction force?





$$\boxed{N = Mg\cos\theta}$$

$$V = \omega R$$

$$\boxed{a = \alpha R}$$

$$\boxed{I_{COM}\,\alpha = R F_f}$$

$$\boxed{Ma = Mg\sin\theta - F_f}$$

$$I\frac{a}{R} = R F_f \Rightarrow F_f = \frac{Ia}{R^2}$$

$$Ma = Mg\sin\theta - \frac{Ia}{R^2}$$

$$a = \left(\frac{M}{M + \frac{I}{R^2}}\right) g \sin\theta$$

$$a\left(M + \frac{I}{R^2}\right) = Mg\sin\theta$$

$$\boxed{F_f = \frac{IM}{I + MR^2}\, g\sin\theta}$$

static friction

$$F_f \leq \mu_s N$$

$$\frac{I\cancel{M}}{I + MR^2}\cancel{g}\sin\theta \leq \mu_s \cancel{M}\cancel{g}\cos\theta$$

$$\boxed{\frac{I}{(I + MR^2)}\tan\theta \leq \mu_s}$$

$$I = \frac{2}{5}MR^2$$

Rotation

θ, ω, α

$\tau = I\alpha$

$\vec{\tau} = \vec{r} \times \vec{p}$

$I = \int r^2\, dm$

$\left(I_d = I_{cm} + Md^2\right)$

$K = \frac{1}{2} I \omega^2$

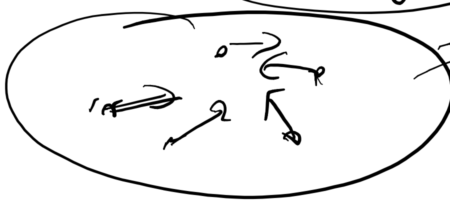
$K_r = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$

Rolling without slipping.

Momentum Conservation

$$\vec{P} = m\vec{v} \Longrightarrow \frac{d\vec{P}}{dt} = \vec{F} \Longrightarrow \Delta\vec{p} = \underbrace{\int \vec{F}\,dt}_{\text{Impulse}}$$

closed system



$\vec{P}_{total} = m_1\vec{v}_1 + m_2\vec{v}_2 + \dots$ is conserved.

$$\vec{P} = \underset{\sum_i m_i}{M}\,\vec{v}_{com} \qquad \vec{r}_{com} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots}{M}$$

for a closed system $\vec{v}_{com}$ is constant.

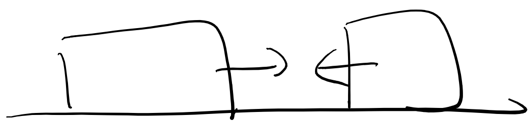
collision

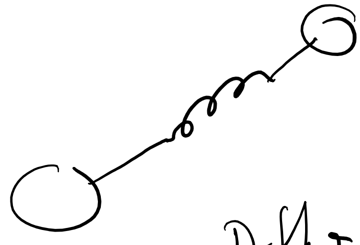
$\Longrightarrow$ Momentum is conserved

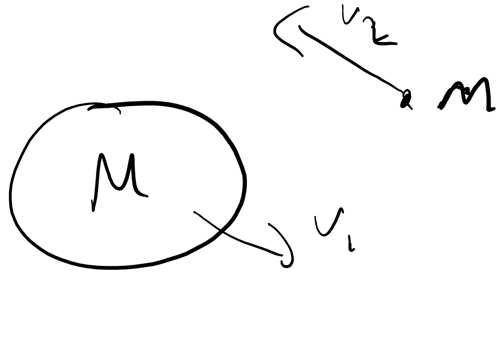
Energy conserved — Elastic

(Inelastic collision)

Objects stick together — Completely Inelastic







Different kinds of collision.

Energy conservation

$$\Delta E = W_{n.c} \rightarrow W = \vec{F}\cdot\vec{d} \quad \text{constant}$$

$$= \int_i^f \vec{F}\cdot d\vec{l}$$

$$E_f - E_i \longrightarrow K = \frac{1}{2}mv^2 \left(+\frac{1}{2}I\omega^2\right)$$

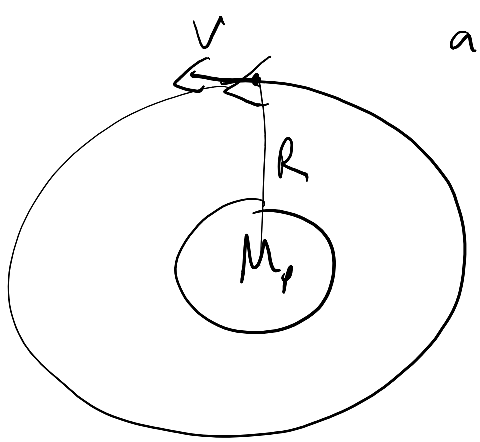
$$U_g = mgh \simeq -G\frac{M_e m}{r}$$

Grav. pot.

$$U_s = \frac{1}{2}k\Delta^2$$

Gravitation $F_G = G\frac{m_1 m_2}{r^2}$

attractive force



$$\cancel{m}\frac{v^2}{\cancel{R}} = F_G = G\frac{M_p \cancel{m}}{R^{\cancel{2}}}$$

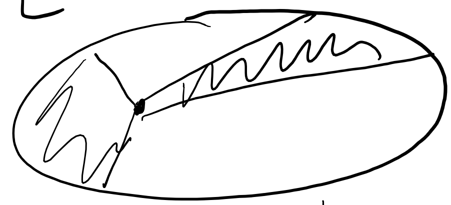
$$v = \sqrt{G\frac{M_p}{R}}$$

$$T = \frac{2\pi R}{\sqrt{G\frac{M_p}{R}}} = \frac{2\pi}{\sqrt{G M_p}}R^{3/2}$$

$$T = \frac{2\pi R}{\sqrt{G \frac{m_p}{R}}} = \frac{2\pi}{\sqrt{G m_p}} R^{3/2}$$

$$\frac{R^3}{T^2} \Longrightarrow \text{constant}$$

[Keplers Laws]



- Ellipses
- Angular momentum.