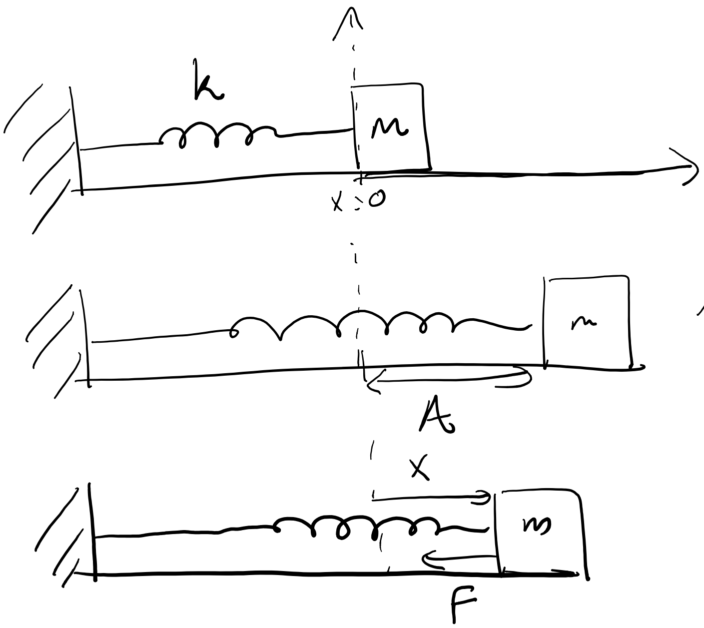
Oscillations



let the system go

$F = -kx$

$$F = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

By inspection $\quad x(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t)$

$$\frac{dx}{dt} = v(t) = \omega C_1 \cos(\omega t) - \omega C_2 \sin(\omega t)$$

$$\frac{d^2x}{dt^2} = -\omega^2 C_1 \sin(\omega t) - \omega^2 C_2 \cos(\omega t)$$

$$= -\omega^2 \left[C_1 \sin(\omega t) + C_2 \cos(\omega t) \right]$$

$$-\omega^2 \left[\cancel{C_1 \sin(\omega t) + C_2 \cos(\omega t)}\right] = -\frac{k}{m}\left[\cancel{C_1 \sin(\omega t) + C_2 \cos(\omega t)}\right]$$

$$\longrightarrow \; -\omega^2 \left[\cancel{C_1 \sin(\omega t) + C_2 \cos(\omega t)} \right] = -\frac{k}{m} \left[\cancel{C_1 \sin(\omega t) + C_2 \cos(\omega t)} \right]$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$ (Angular) Frequency of oscillations

Now I know the solution is

$$x(t) = C_1 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

$C_1 = ?$
$C_2 = ?$

$$x(t=0) = A$$

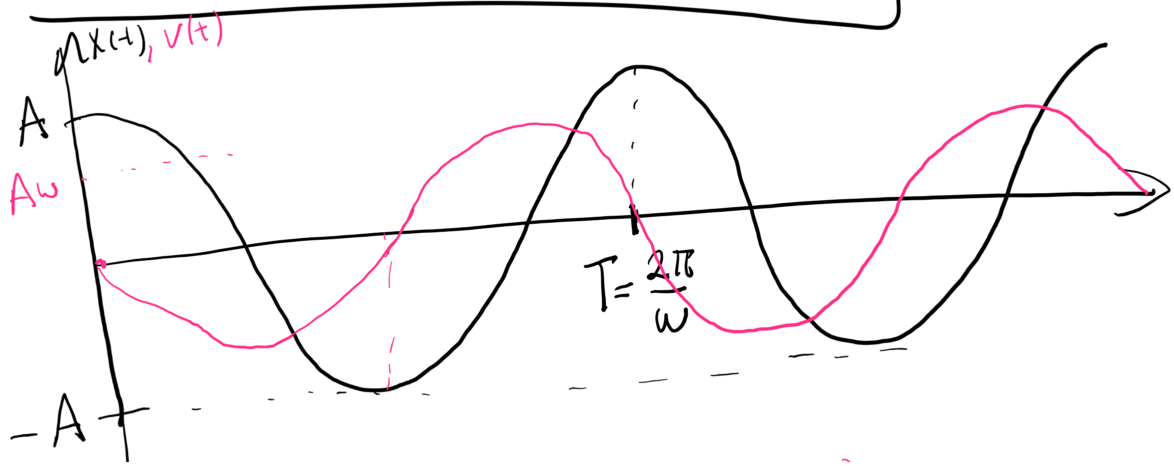
$$v(t=0) = 0$$

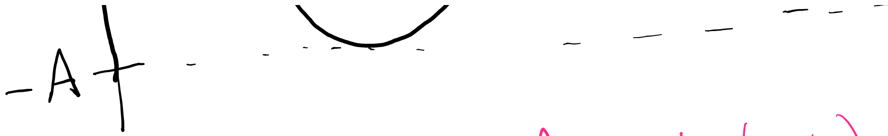
$$C_1 \underbrace{\sin(0)}_{0} + C_2 \underbrace{\cos(0)}_{1} = A \Rightarrow \boxed{C_2 = A}$$

$$v(t=0) = \omega C_1 \underbrace{\cos(0)}_{1} - \omega C_2 \underbrace{\sin(0)}_{0} = 0$$

$$\boxed{C_1 = 0}$$

$$\boxed{x(t) = A \cos\left(\sqrt{\frac{k}{m}}\, t\right)} \qquad \omega = \sqrt{\frac{k}{m}}$$





$$v(t) = -A\omega \sin(\omega t)$$

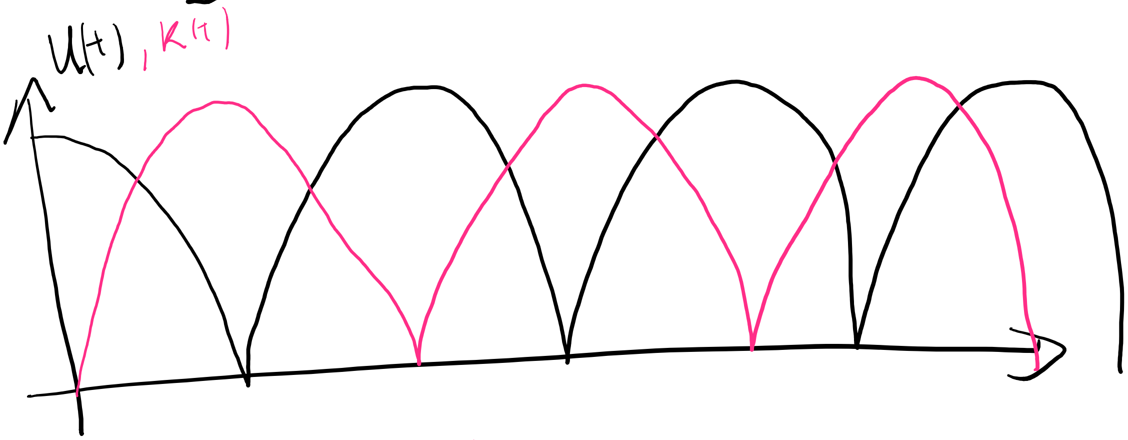
$$K(t) = \frac{1}{2} m v^2(t) = \frac{1}{2} m A^2 \overset{k/m}{\omega^2} \sin^2(\omega t) = \frac{1}{2} k A^2 \sin^2(\omega t)$$

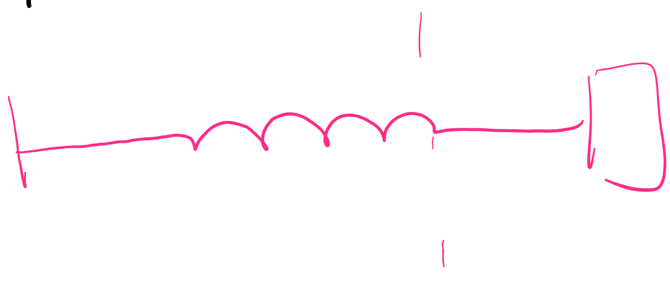
$$U(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k A^2 \cos^2(\omega t)$$

$$E = K(t) + U(t)$$

$$= \frac{1}{2} k A^2 \underline{\sin^2(\omega t)} + \frac{1}{2} k A^2 \underline{\cos^2(\omega t)}$$

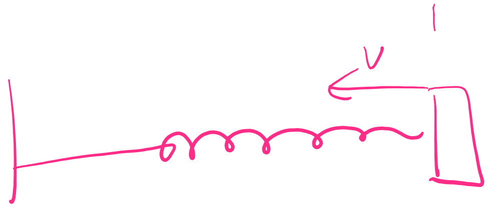
$$= \frac{1}{2} k A^2 \left(\sin^2(\omega t) + \cos^2(\omega t)\right) = \frac{1}{2} k A^2$$





$U = \frac{1}{2} k A^2$

$K = 0$



$U = 0$

$K = \frac{1}{2} k A^2$

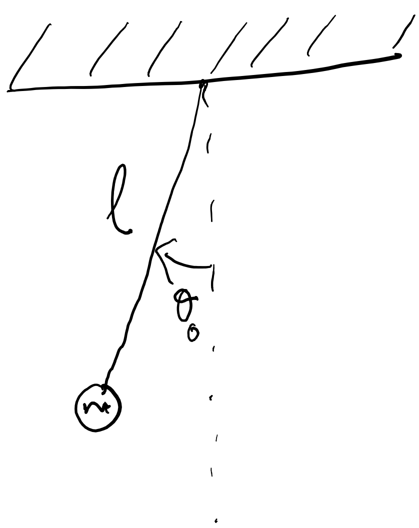
Energy oscillates between kinetic and potential forms.

$$\omega = \sqrt{\frac{k}{m}}$$, is independent of Amplitude!!

If T (or ω) of an oscillator is independent of its amplitude it is called a Harmonic Oscillator.

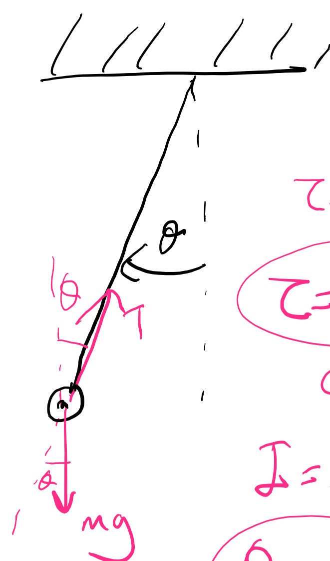
Harmonic oscillations are good models for almost all oscillators when the amplitude is small! approximations

Ex



gb

$\theta(t) = ?$



$\tau = \vec{r} \times \vec{F}$

$\tau = lmg\sin\theta$

c.c.w.

$I = ml^2$

θ c.w

$\tau = I\alpha$

$$lmg\sin\theta = -ml^2\frac{d^2\theta}{dt^2}$$

$\ell mg \sin\theta = -m\ell \frac{d^2\theta}{dt^2}$ $\quad$ (θ c.w)

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin(\theta)$$

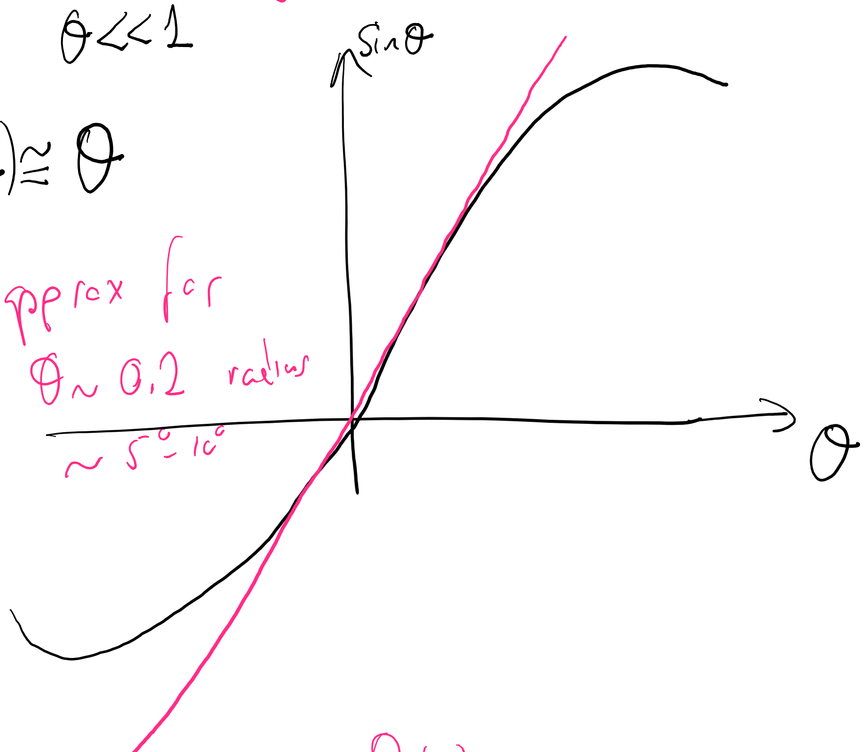
$\theta \ll 1$

$\sin(\theta) \cong \theta$

good approx for

$\theta \sim 0.2$ radius

$\sim 5^\circ - 10^\circ$



$$\frac{d^2\theta(t)}{dt^2} = -\frac{g}{\ell}\cancel{\sin\theta}\ \theta(t)$$ approximation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta(t)$$

$$\theta(t) = C_1\sin(\omega t) + C_2\cos(\omega t)$$

$$\frac{d^2\theta}{dt^2}(t) = -\omega^2 C_1\sin(\omega t) - \omega^2 C_2\cos(\omega t) = -\omega^2\theta(t)$$

$$-\omega^2\cancel{\theta(t)} = -\frac{g}{\ell}\cancel{\theta(t)}$$

$$\omega = \sqrt{\frac{g}{\ell}} \qquad \text{Frequency of "small" oscillations}$$

$$T = \frac{2\pi}{\sqrt{\frac{g}{\ell}}} = 2\pi\sqrt{\frac{\ell}{g}}$$

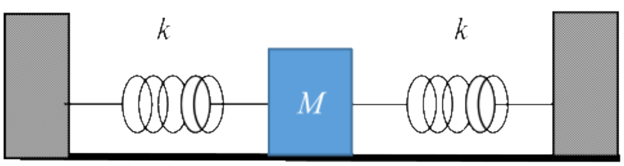
$$\theta(t=0) = \theta_0 = C_2$$

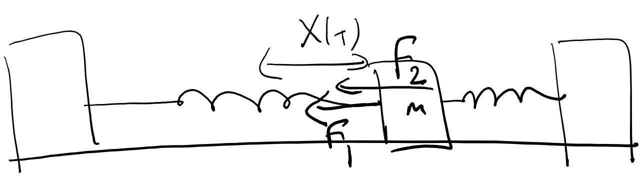
$$\frac{d\theta}{dt}(t=0) = 0 = -\omega C_1 \Rightarrow C_1 = 0$$

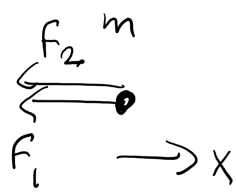
$$\theta(t) = \theta_0 \cos(\omega t) \qquad \omega = \sqrt{\frac{g}{\ell}}$$

Phys 101- Instructor: M. Özgür Oktel – 2016 QUIZ-25

A mass ***M*** on a frictionless table is connected to two walls with two identical springs of spring constant ***k***. The springs are at their relaxed length in equilibrium. Find the (angular) frequency of small oscillations of the mass.







$$|F_1| = kx$$
$$|F_2| = kx$$

$$ma = m\frac{d^2x}{dt^2} = -|F_1| - |F_2|$$
$$= -2kx$$

$$m\frac{d^2x}{dt^2} = -2kx$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{2k}{m}x}$$

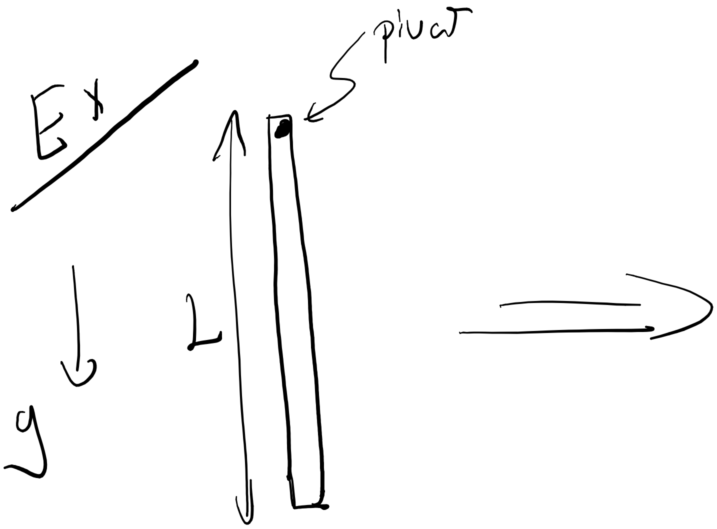
$\omega = ?$

$$x = C_1\cos(\omega t) + C_2\sin(\omega t)$$

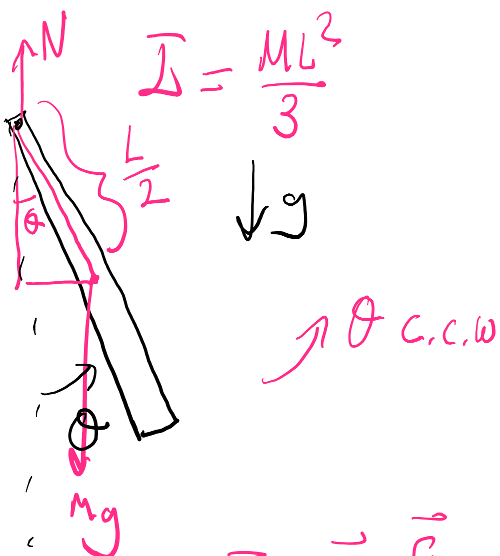
$$\frac{d^2x}{dt^2} = -\omega^2\left(C_1\cos(\omega t) + C_2\sin(\omega t)\right)$$

$$\omega^2 = \frac{2k}{m} \Longrightarrow \boxed{\omega = \sqrt{\frac{2k}{m}}}$$

Ex



$I = \frac{ML^2}{3}$



$\omega = ?$
small oscillations

$$\tau = I\alpha$$

$$\tau = \vec{r}\times\vec{F}$$

$$= |\vec{r}||\vec{F}|\sin\theta$$

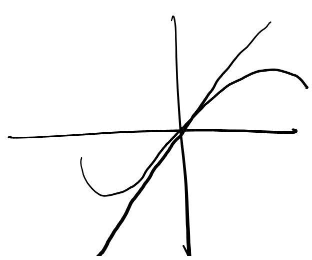
$$= \frac{L}{2}\sin\theta\, Mg$$

$$\tau = \frac{MgL}{2}\sin\theta \quad \boxed{\text{c.w}}$$

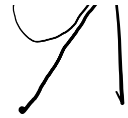
$$-\frac{Mg\cancel{L}}{2}\sin\theta = \frac{M L^{\cancel{2}}}{3}\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\sin\theta$$

Small oscillation $\quad \sin\theta \simeq \theta$



Small oscillation $\sin\theta \simeq \theta$



$$\frac{d^2\theta}{dt^2} \simeq -\frac{3}{2}\frac{g}{L}\theta$$

$$\theta(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$$

$$\frac{d^2\theta}{dt^2} = -\omega^2\,\theta(t)$$

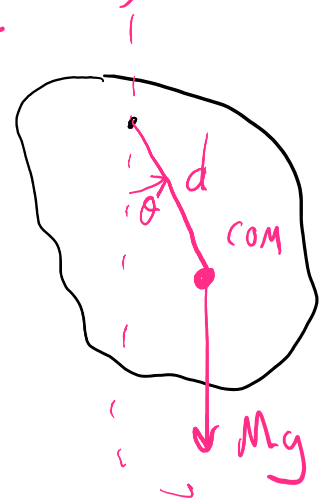
$$\omega^2 = \frac{3}{2}\frac{g}{L} \implies \boxed{\omega = \sqrt{\frac{3}{2}\frac{g}{L}}}$$

$$\sqrt{\frac{m/s^2}{m}} = \frac{1}{s}$$

"physical pendulum"

I_{com}, d, M

$I = I_{com} + Md^2$



$\theta \circlearrowleft$ c.c.w

$$\tau = \vec{r}\times\vec{F} = dMg\sin\theta \quad \text{c.w.}$$

$$I\alpha = \tau$$

$$(I_{com} + Md^2)\frac{d^2\theta}{dt^2} = -dMg\sin\theta$$

$(I_{com}+Md^2)\frac{d^2\theta}{dt^2} = \ldots$

$$\frac{d^2\theta}{dt^2} = -\frac{dMg}{(I_{com}+Md^2)}\sin\theta$$

small oscillations $\sin\theta \simeq \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{dMg}{(I_{com}+Md^2)}\theta$$

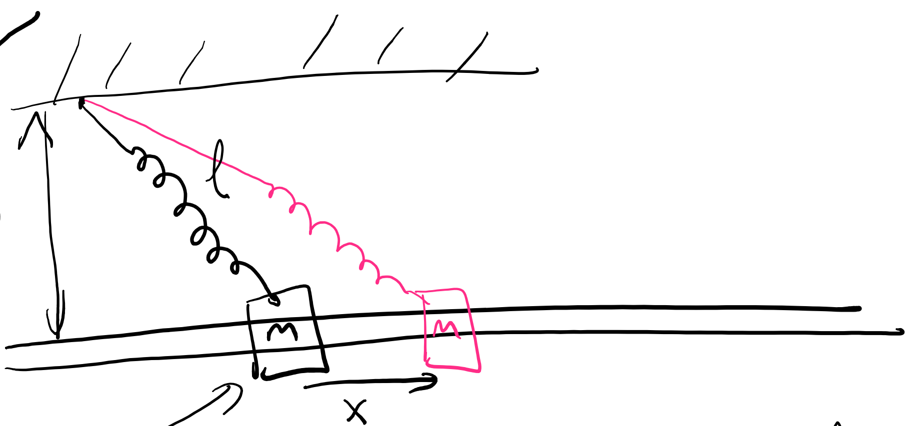
$$\theta(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$$

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

$$\boxed{\omega = \sqrt{\frac{dMg}{I_{com}+Md^2}}}$$

$$[\omega] = \sqrt{\frac{\cancel{m}\,\cancel{kg}\,\cancel{m}/s^2}{\cancel{kg}\,\cancel{m^2}}} = \frac{1}{s} \checkmark$$

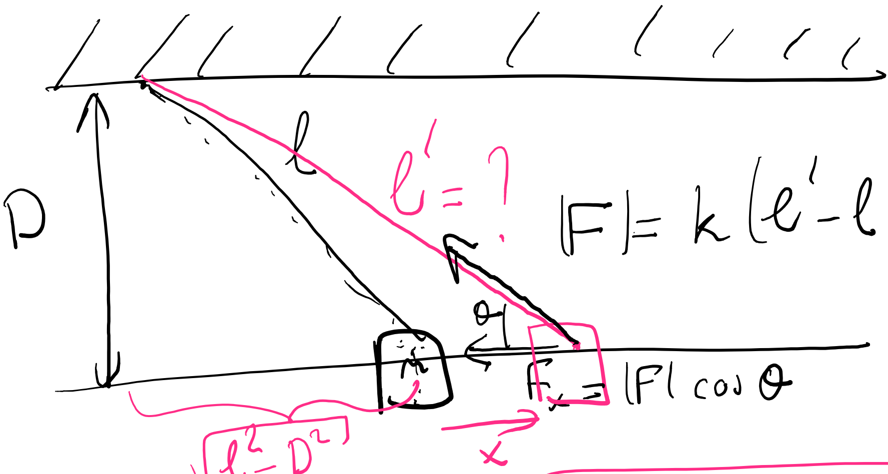
Ex



bead on a frictionless wire

relaxed length of spring is l.

Find the frequency of small oscillations



$$|F| = k(\ell' - \ell)$$

$$\cos\theta = \frac{\sqrt{\ell^2 - D^2} + x}{\ell'}$$

$$\ell' = ? \Rightarrow \ell' = \sqrt{D^2 + \left(\sqrt{\ell^2 - D^2} + x\right)^2}$$

$$|F_x| = k(\ell' - \ell)\frac{\sqrt{\ell^2 - D^2} + x}{\ell'}$$

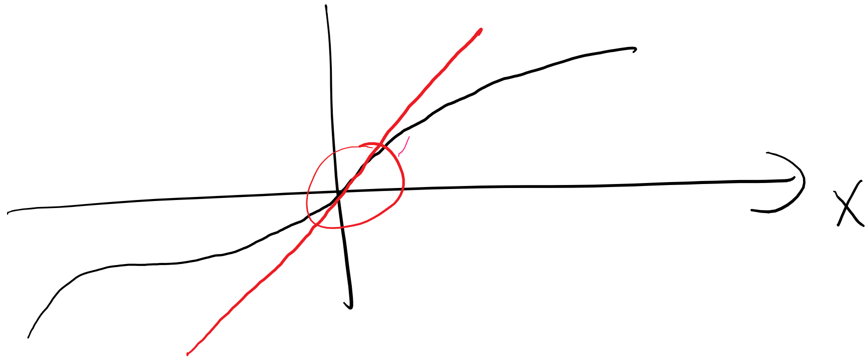
$$m\frac{d^2x}{dt^2} = -k(\ell' - \ell)\frac{\sqrt{\ell^2 - D^2} + x}{\ell'}$$

$$m\frac{d^2x}{dt^2} = \underbrace{-k\left[\sqrt{D^2 + \left(\sqrt{\ell^2 - D^2} + x\right)^2} - \ell\right]\frac{\sqrt{\ell^2 - D^2} + x}{\sqrt{D^2 + \left(\sqrt{\ell^2 - D^2} + x\right)^2}}}_{= f(x)}$$

$$\ell' - \ell = 0 \text{ if } x = 0$$

$$f(x=0) = 0$$

$f(x)$ ← linear approximation!



$$f(x) \underset{x \ll 1}{\simeq} f(x=0) + \left.\frac{df}{dx}\right|_{x=0} x + \cancel{\ldots}$$

neglect!

Taylor series

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} (x-x_0) + \frac{1}{2!} \left.\frac{d^2f}{dx^2}\right|_{x_0} (x-x_0)^2 \ldots$$

$$f(x) = -k\left[\sqrt{D^2 + \left(\sqrt{\ell^2-D^2}+x\right)^2} - \ell\right] \frac{\sqrt{\ell^2-D^2}+x}{\sqrt{D^2+\left(\sqrt{\ell^2-D^2}+x\right)^2}}$$

$$\frac{df}{dx} = -k\left[\frac{1}{2}\,\frac{2\left(\sqrt{\ell^2-D^2}+x\right)}{\sqrt{\quad}}\right]\frac{\sqrt{\ell^2-D^2}+x}{\sqrt{\quad}}$$

$$-k\left[\sqrt{\quad} - \ell\right] \frac{\sqrt{\quad} - \left(\sqrt{\ell^2-D^2}+x\right)\frac{1}{2}\,\frac{2\left(\sqrt{\ell^2-D^2}+x\right)}{\sqrt{\quad}}}{\left(\sqrt{\quad}\right)^2}$$

$$\left.\frac{df}{dx}\right|_{x=0} = -k\left[\frac{1}{\cancel{2}}\,\frac{\cancel{2}\sqrt{\ell^2-D^2}}{\ell}\right]\frac{\sqrt{\ell^2-D^2}}{\ell} - \ldots$$

$$-k\,\cancel{\left(\frac{\ell}{\ell} - \frac{\ell}{c}\right)} \ldots\ldots$$

$$= -k\,\frac{\ell^2-D^2}{\ell^2}$$

Now

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\,\frac{\ell^2-D^2}{\ell^2}\,x$$

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$$

$$\boxed{\omega = \sqrt{\frac{k}{m}\,\frac{\ell^2-D^2}{\ell^2}}}$$

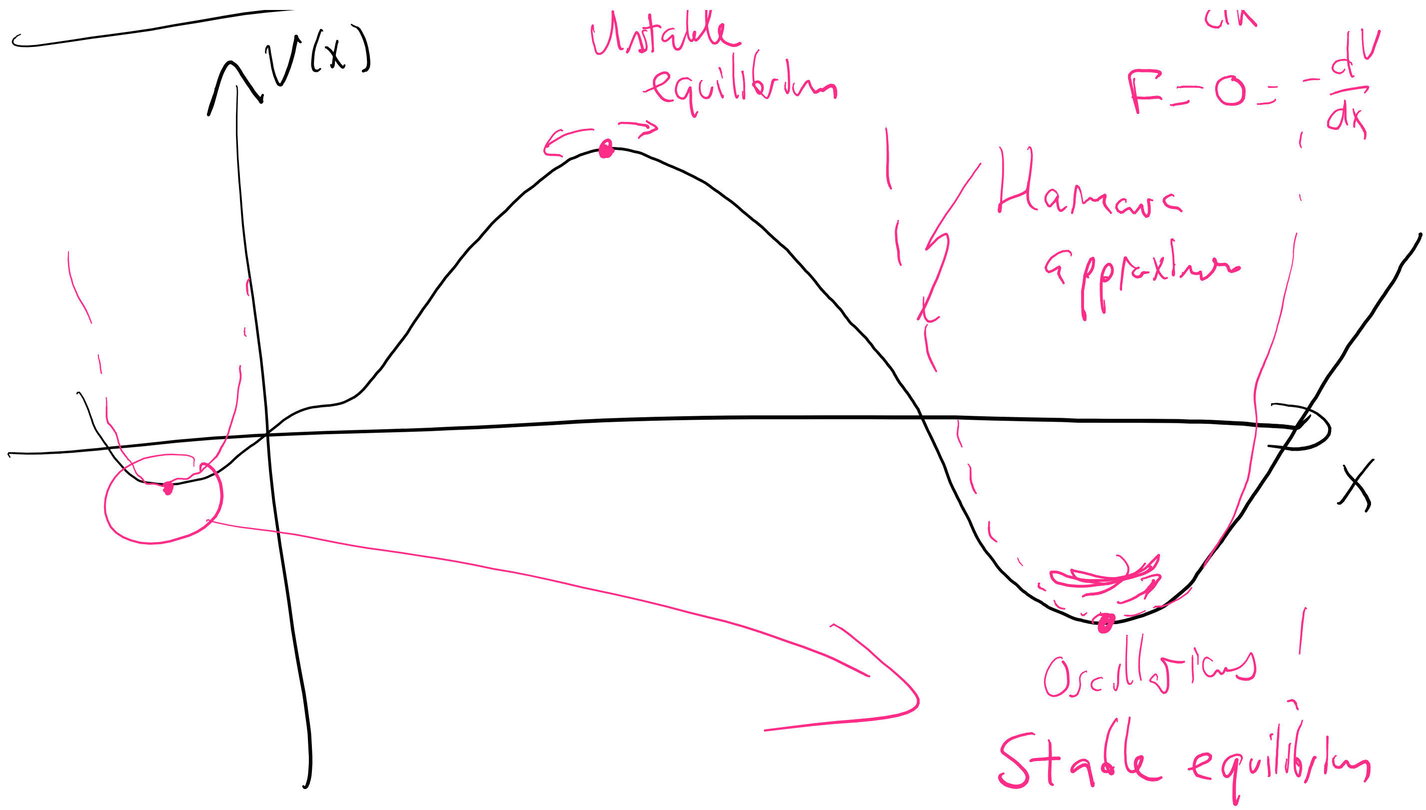
General properties

$V(x)$

Unstable equilibrium

$$F = -\frac{dV(x)}{dx}$$

$$F = 0 = -\frac{dV}{\;}$$



Energy conserving

No friction?

$\Longrightarrow$ Damped oscillations