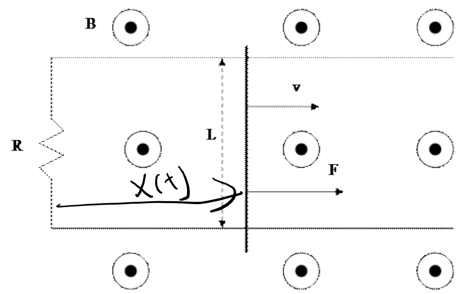
# Week 13 Notes (Apr 18-22)

Wednesday, January 27, 2016 4:48 PM

Quiz 24
PHYSICS 102- Instructor: M. Özgür OKTEL- Spring 2016

A rod of length L can slide without friction on rails as shown in the figure. The rod and rails have negligible resistance but are connected to a resistor with resistance R. If the rod is moving with velocity v as shown in the figure.
a) Find the direction and magnitude of the current on the resistor.
b) To keep the rod moving with velocity v what is the force that must be applied to it? Find F.



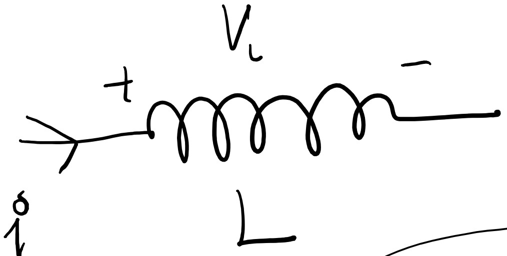

$$|\mathcal{E}| = \left|\frac{d\Phi}{dt}\right| = BL\frac{dx}{dt} = BvL$$

a) $$i = \frac{\mathcal{E}}{R} = \boxed{\frac{BvL}{R}}$$

b) $$\vec{F} = i\vec{L}\times\vec{B} \Rightarrow iLB = |\vec{F}|$$

$$\boxed{\frac{B^2L^2v}{R} = |\vec{F}|}$$

# Inductors




$$V_L = L\frac{di}{dt} = \frac{d}{dt}(Li) = \frac{d}{dt}(\Phi_B)$$

$$P = Vi = \underbrace{L\frac{di}{dt}}\, i = \frac{1}{2}\frac{d}{dt}(Li^2) = \frac{d}{dt}\underbrace{\left(\frac{1}{2}Li^2\right)}$$

$$\frac{d}{dt}(i^2) = 2i\frac{di}{dt}$$

$$\frac{d}{dt}(i^2) = 2\, i \frac{di}{dt}$$

$$U_L = \frac{1}{2} L i^2$$

potential energy stored in an inductor

$$U_C = \frac{1}{2} C V^2 \qquad u_E = \frac{1}{2} \varepsilon_0 E^2$$

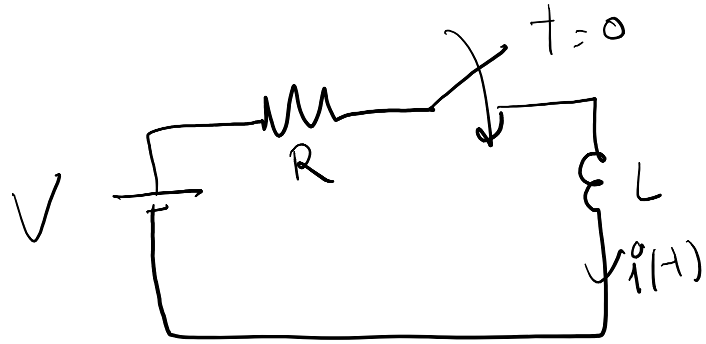
energy density

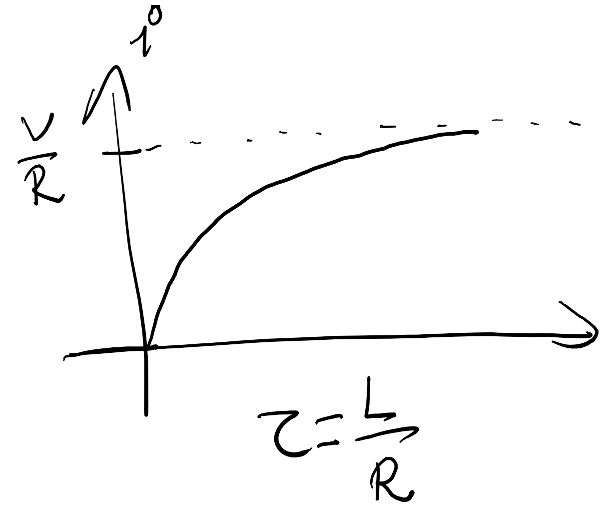
$$\boxed{u_B = \frac{1}{2\mu_0} B^2} \Longrightarrow$$

energy density of the magnetic field.

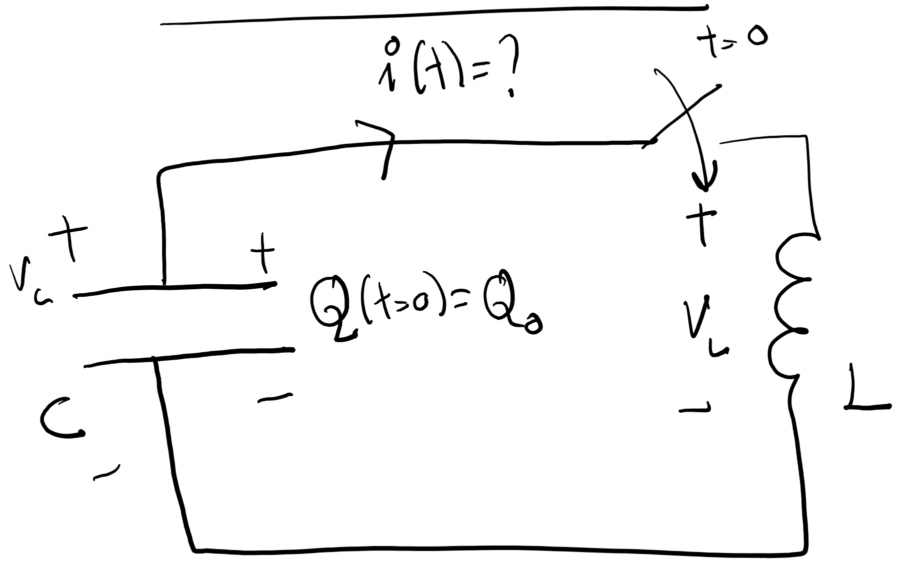
R | C | L
--- | --- | ---
$V = R i$ | $i = C \frac{dV}{dt}$ | $V = L \frac{di}{dt}$
Dissipate | Store energy | Stored energy
 | $\boxed{U_C = \frac{1}{2} C V^2}$ | $\boxed{U_L = \frac{1}{2} L i^2}$

Recall we solved RL




# LC circuit




Initial energy

$$E_i = \frac{1}{2}\frac{Q_0^2}{C}$$

will be conserved.

$$V_L = L\frac{di}{dt} = V_c = \frac{Q(t)}{C}$$

$$i(t) = -\frac{dQ}{dt}$$

$$L\frac{di}{dt} = \frac{Q(t)}{C} \qquad\qquad i = -\frac{dQ}{dt} \;\Big\downarrow \frac{d}{dt}$$

$dt \qquad C \qquad dt \qquad \frac{d}{dt}$

$$\frac{di}{dt} = -\frac{d^2}{dt^2}Q$$

$$-L\frac{d^2Q}{dt^2} = \frac{Q}{C}$$

$$\boxed{\frac{d^2}{dt^2}\left(Q(t)\right) = -\frac{1}{LC}Q(t)}$$

Only 2 functions can satisfy this $\quad \cos(\Omega t), \sin(\Omega t)$

$$Q(t) = A\cos(\Omega t) + B\sin(\Omega t)$$

$$\frac{dQ}{dt} = -\Omega A\sin(\Omega t) + \Omega B\cos(\Omega t)$$

$$\frac{d^2Q}{dt^2} = -\Omega^2 A\cos(\Omega t) - \Omega^2 B\sin(\Omega t)$$

$$\frac{d^2Q}{dt^2} = -\Omega^2 Q(t) \iff \frac{d^2Q}{dt^2} = -\frac{1}{LC}Q(t)$$

$$\Omega^2 = \frac{1}{LC}$$

$$\boxed{\Omega = \frac{1}{\sqrt{LC}}} \qquad \omega_0$$

$$Q(t) = A\cos\left(\frac{t}{\sqrt{LC}}\right) + B\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$Q(t=0) = \boxed{Q_0 = A} \Rightarrow \boxed{A = Q_0}$$

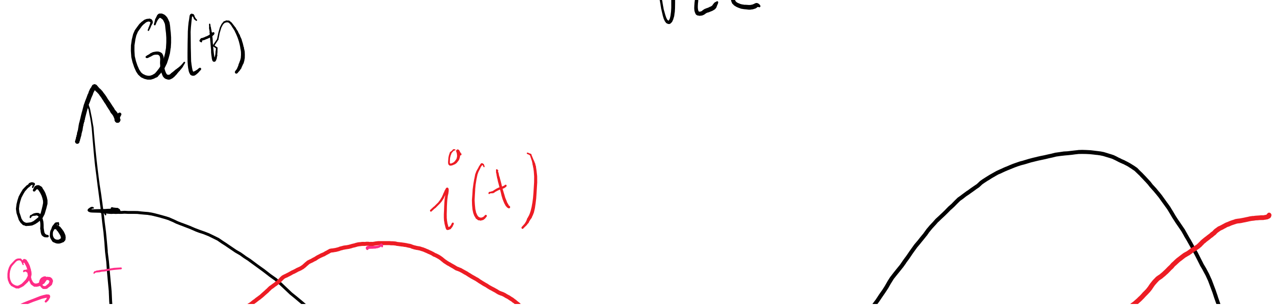
$$i(t) = -\frac{dQ}{dt} = -\frac{1}{\sqrt{LC}}A\sin\left(\frac{t}{\sqrt{LC}}\right) + \frac{B}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

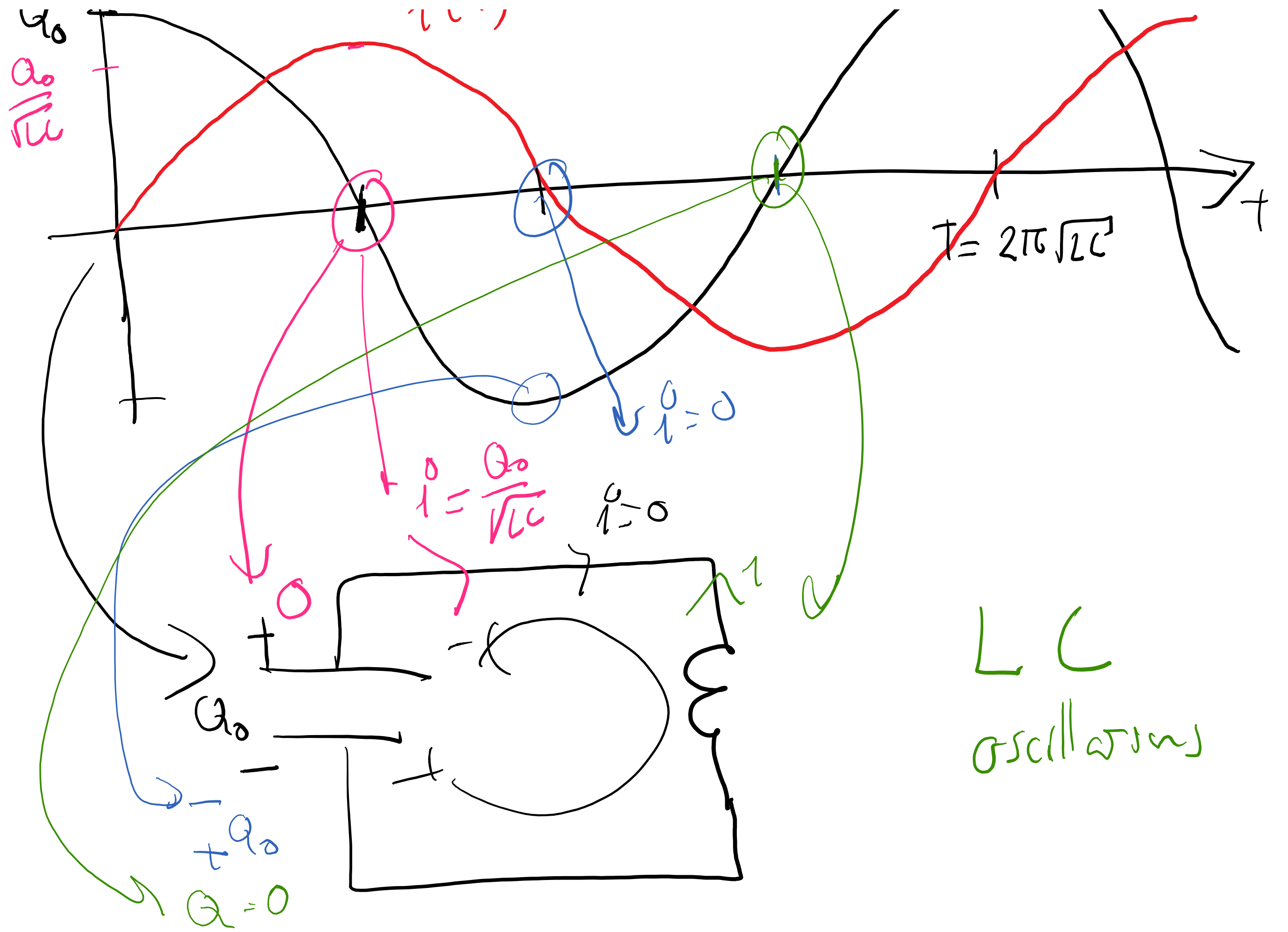
$$i(t=0) = 0 = \frac{B}{\sqrt{LC}} \Rightarrow \boxed{B = 0}$$

$$Q(t) = Q_0\cos\left(\frac{t}{\sqrt{LC}}\right) \qquad \Omega = \frac{1}{\sqrt{LC}}$$

$$T = \frac{2\pi}{\Omega}$$

$$i(t) = -\frac{d}{dt}Q(t) = \frac{Q_0}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)$$






$$Q(t) = Q_0 \cos\left(\frac{t}{\sqrt{LC}}\right) \qquad i(t) = \frac{Q_0}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

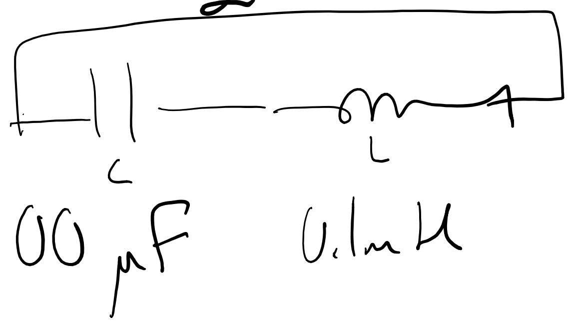
$$U_C = \frac{1}{2}\frac{Q(t)^2}{C} \qquad U_L = \frac{1}{2} L\, i(t)^2$$

$$= \frac{1}{2}\frac{1}{C} Q_0^2 \cos^2\left(\frac{t}{\sqrt{LC}}\right) \qquad = \frac{1}{2}\cancel{L}\frac{Q_0^2}{\cancel{L}C}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$U_C + U_L = \frac{1}{2}\frac{Q_0^2}{C}\left(\cos^2\left(\frac{t}{LC}\right) + \sin^2\left(\frac{t}{LC}\right)\right)$$

$$U_c + U_L = \frac{1}{2}C\underbrace{\left(\cos^2 \frac{t}{\sqrt{LC}} + \sin^2 \frac{t}{\sqrt{LC}}\right)}_{1}$$

$$= \frac{1}{2}\frac{Q_0^2}{C}$$

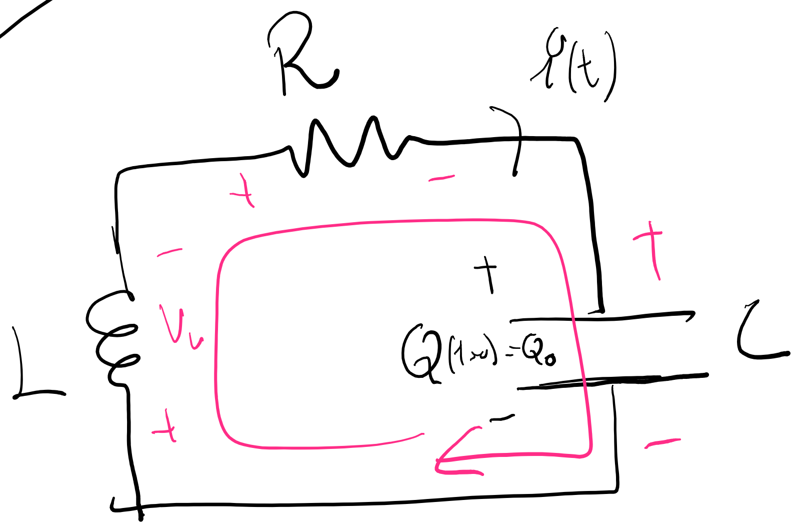



$$T = 2\pi\sqrt{LC}$$

$$= 2\pi\sqrt{10^{-4}\,10^{-4}} = 2\pi\,10^{-4}\,\text{sec}$$

$$\sim 1\ \text{msc}$$

~~Ex~~




$$i(t) = \frac{dQ}{dt}$$

RLC circuit.

$$V_L = L\frac{di}{dt} \qquad V_C = \frac{Q(t)}{C}$$

$$V_R = Ri$$

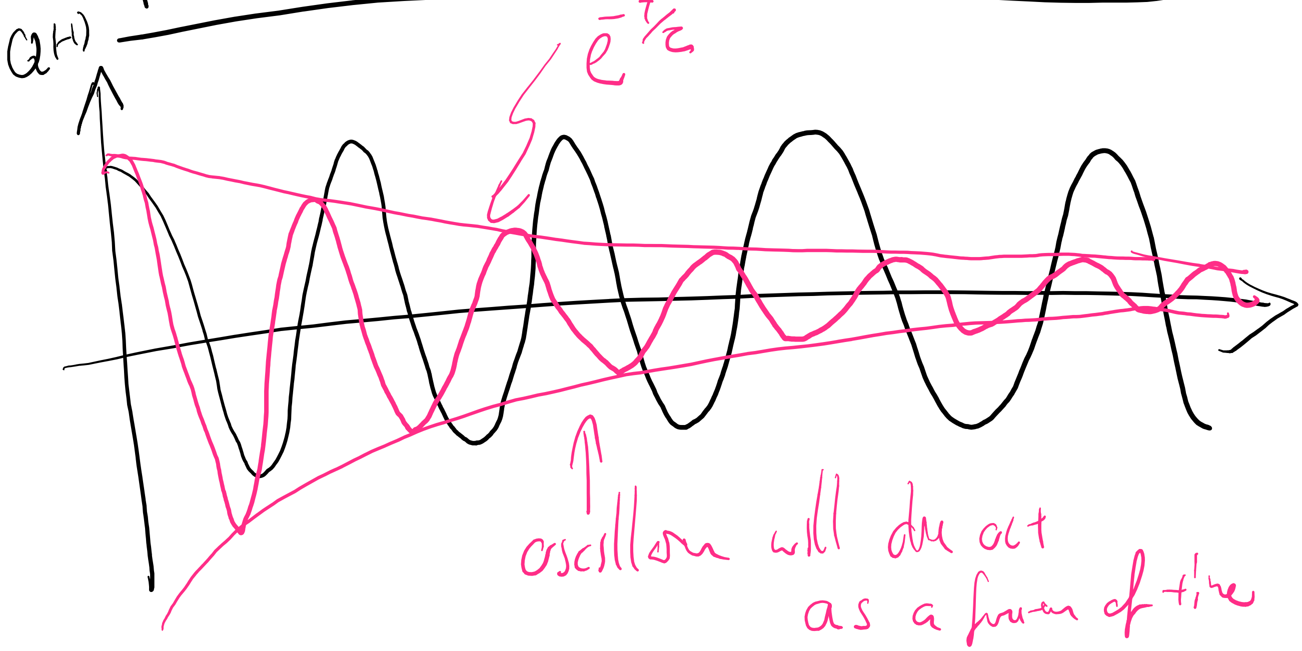
$$V_L + V_R + V_C = 0$$

$$i = \frac{dQ}{dt}$$

$$i = \frac{dQ}{dt}$$

$$L \frac{di}{dt} + R i + \frac{Q}{C} = 0$$

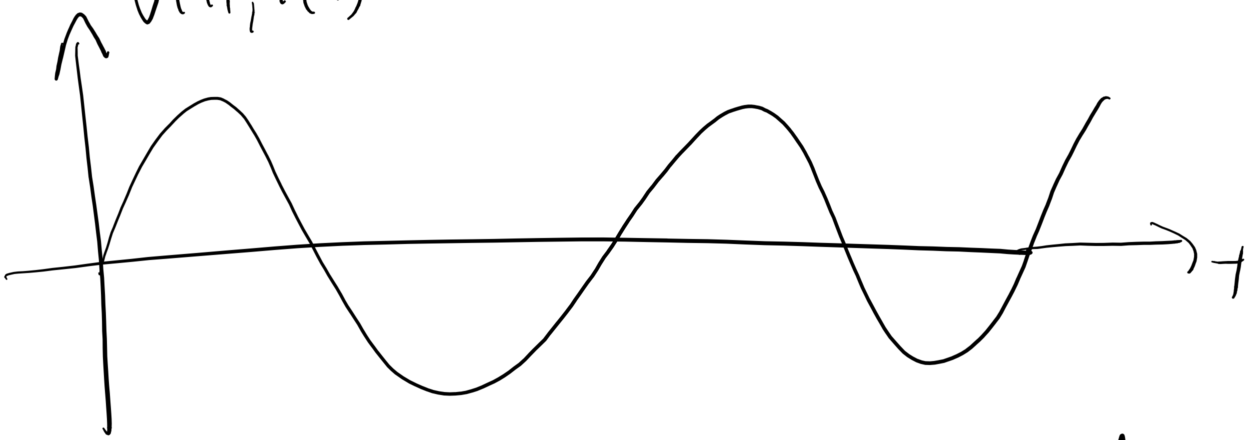
$$\boxed{\frac{d^2}{dt^2} Q + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} = 0}$$



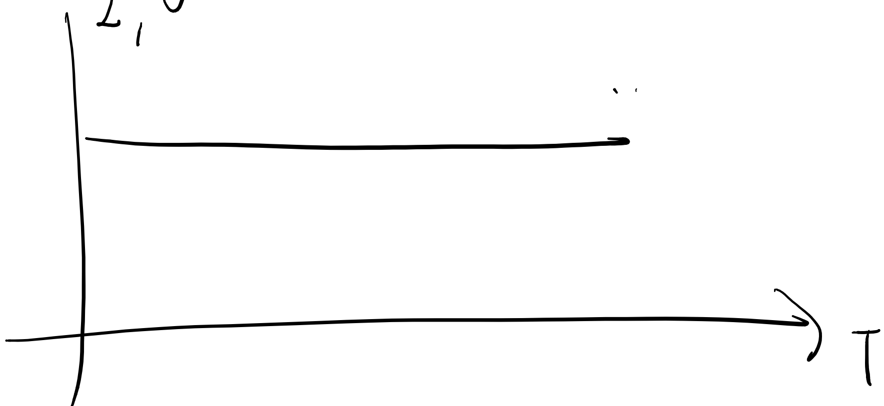

$$Q(t) = Q_0 \cos(\Omega t)\, e^{-t/\tau}$$

$$\Omega \neq \frac{1}{\sqrt{LC}} = \Omega_0 \qquad \boxed{\tau = \frac{L}{R}}$$

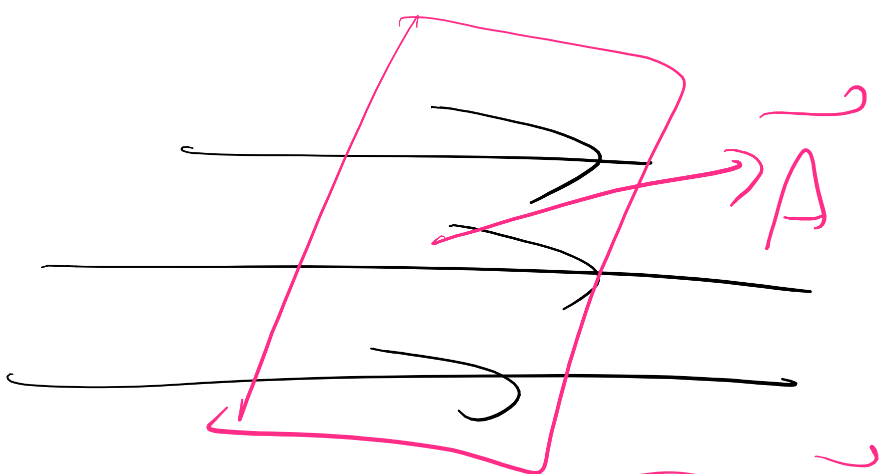
# Alternating Current (AC)




Most of our power systems rely on AC



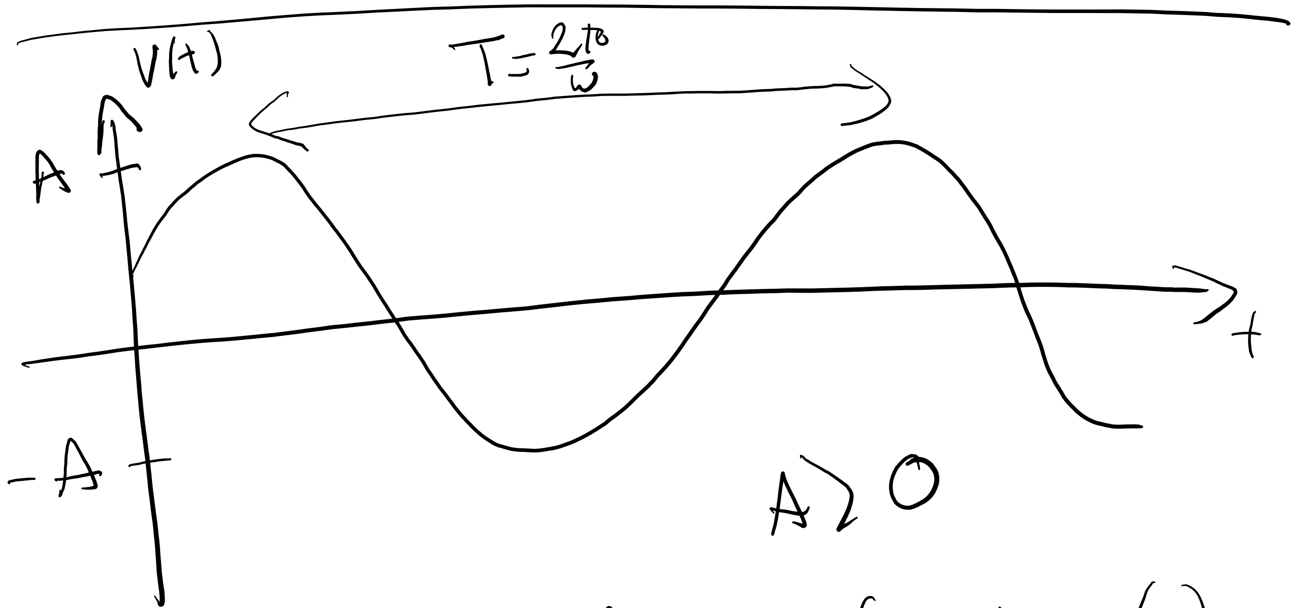

(DC)




$$\Phi = \vec{B}\cdot\vec{A}$$
$$= |\vec{B}||\vec{A}|\cos\theta$$

$$\frac{d\Phi}{dt} = -\omega |\vec{B}||\hat{A}| \sin(\omega t)$$

$$\Phi = |\vec{B}||\hat{A}| \cos \underbrace{\theta}_{\omega t}$$




$A > 0$

$$V(t) = A \cos(\omega t + \phi)$$

Amplitude

Ang. Frequency

$$\nu = \frac{\omega}{2\pi}$$

$$T = \frac{1}{\nu} = \frac{2\pi}{\omega}$$

phase

$$-\pi < \phi < \pi$$

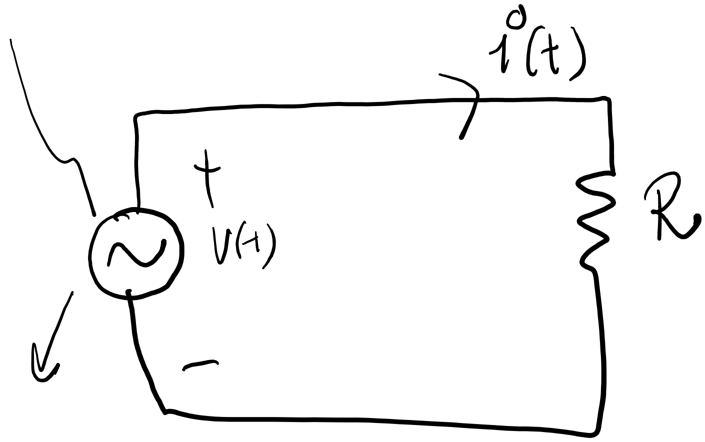
$$\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$$

$$5 \sin(\omega t) = 5 \cos\left(\omega t - \frac{\pi}{2}\right)$$

$A = 5$ $\quad \omega = \omega$ $\quad \boxed{\phi = -\frac{\pi}{2}}$

For AC circuits Phase angle as well as Amplitude are important.

AC voltage source




$$i(t) = \frac{V(t)}{R}$$

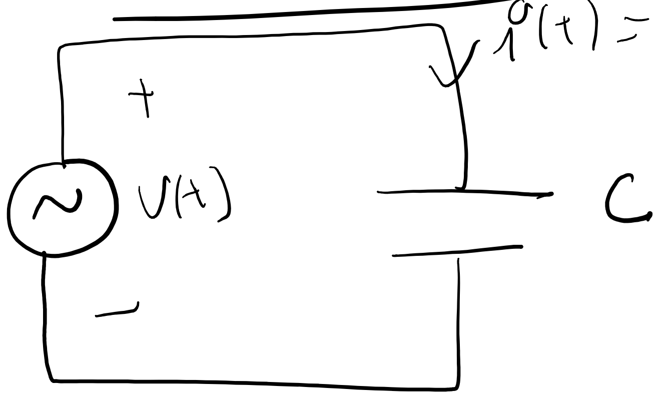
$$V(t) = V_0 \cos(\omega t)$$

$$i(t) = \frac{V_0}{R} \cos(\omega t)$$

$$\boxed{Z = \frac{V_0}{\frac{V_0}{R}} = R}$$

$i(t) = ?$

$V_0$




$$i = C \frac{dV}{dt}$$

$$V(t) = V_0 \cos(\omega t)$$

$$i(t) = C \frac{d}{dt} \left( V_0 \cos(\omega t) \right)$$

$$= -C V_0 \omega \sin(\omega t)$$

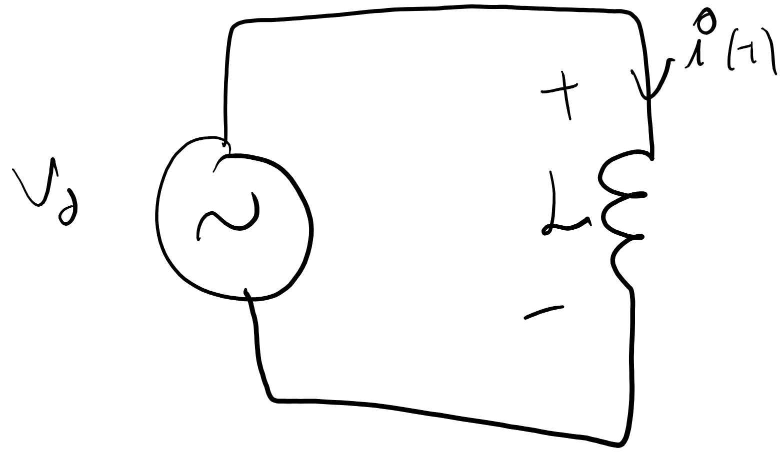
$$= -C V_0 \omega \cos(\omega t - \frac{\pi}{2})$$

$$\boxed{i(t) = C V_0 \omega \cos(\omega t + \frac{\pi}{2}) = C\omega \, V_0 \cos(\omega t + \frac{\pi}{2})}$$

Accross the capacitor <u>i(t) leads V(t)</u> by $\pi/2$

$$\frac{|V|}{|i|} \sim \frac{V_0}{C\omega V_0} = \boxed{\frac{1}{\omega C}} = Z$$

AC circuit $\quad \frac{V_{amp}}{I_{amp}} = Z$ ← impedance




$V(t) = V_0 \cos(\omega t)$

$i(t) = ?$

$V(t) = L \frac{di}{dt}$

$$V_0 \cos(\omega t) = L \frac{d}{dt} i(t)$$

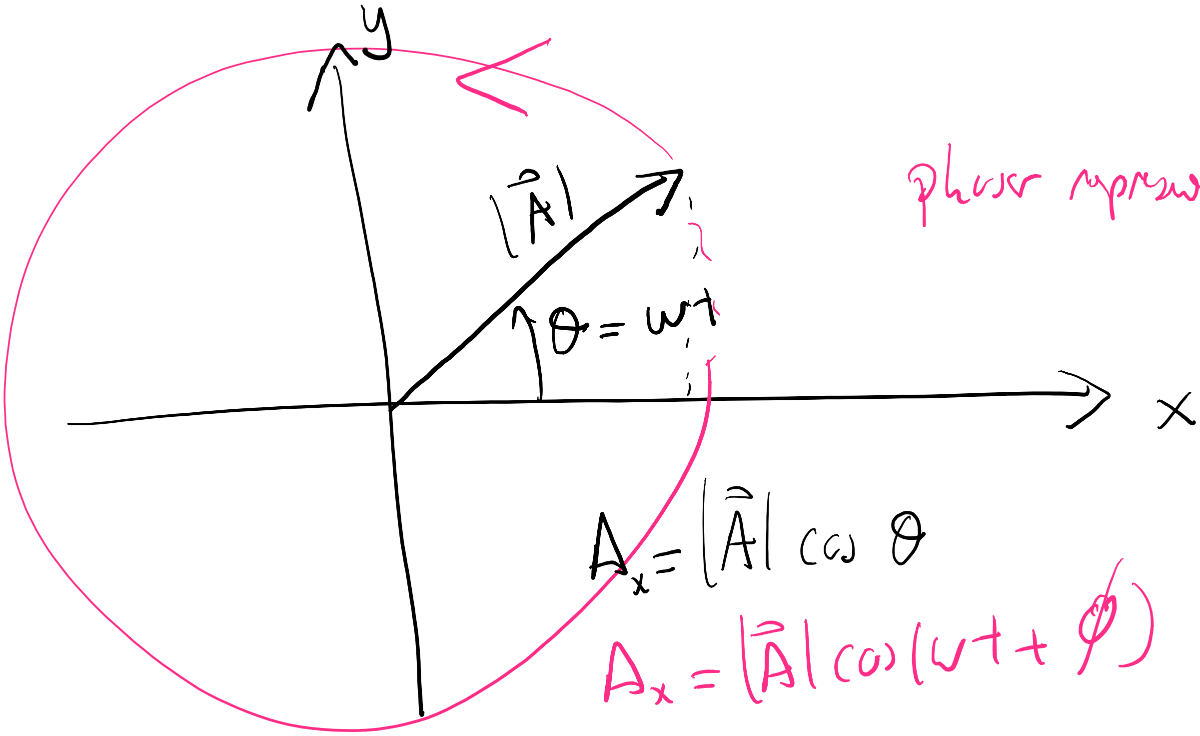
$$\boxed{i(t) = \frac{V_0}{L\omega} \sin(\omega t)}$$

$$\frac{di}{dt} = \frac{V_0}{L\cancel{\omega}} \cancel{\omega} \cos(\omega t) \Rightarrow L\frac{di}{dt} = V_0 \cos(\omega t) \checkmark$$

$$i(t) = \frac{V_0}{L\omega} \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\frac{V_{amp}}{i_{amp}} = \frac{V_0}{\frac{V_0}{L\omega}} = \boxed{L\omega}$$

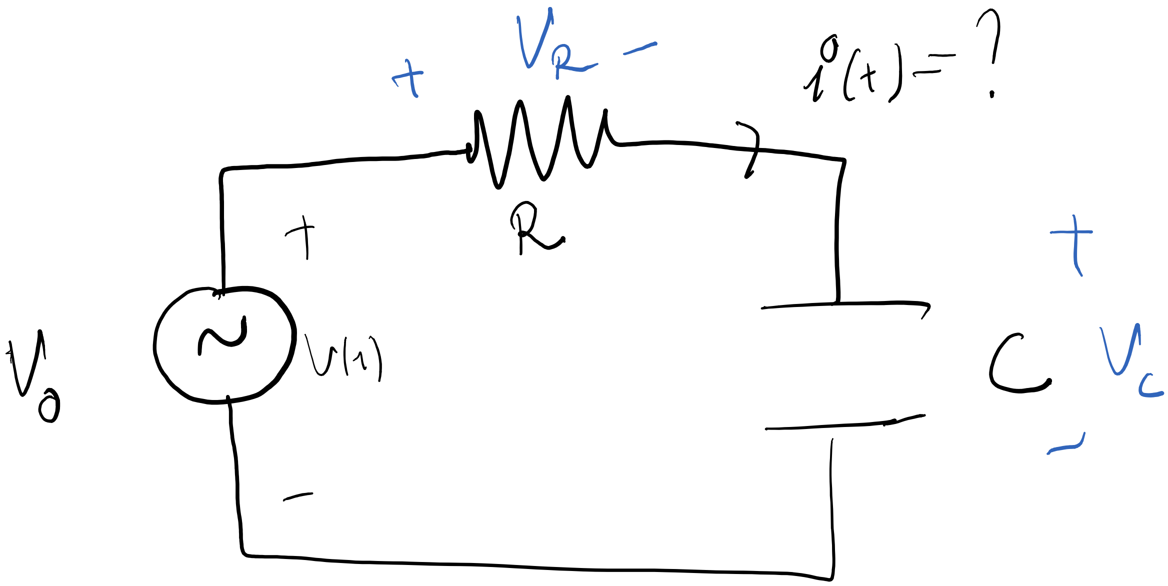
Current across an inductor lags behind the voltage by $\frac{\pi}{2}$




phasor represen

$$A_x = |\vec{A}| \cos\theta$$

$$A_x = |\vec{A}| \cos(\omega t + \phi)$$




$$V(t) = V_0 \cos(\omega t) \qquad i(t) = ?$$

$$V_R(t) = R\, i(t)$$

$$V_c(t) = ? \qquad C \frac{dV_c}{dt} = i(t)$$

Capacitor $|Z| = \frac{1}{\omega C}$ $\quad i(t)$ leads $V(t)$ by $\frac{\pi}{2}$

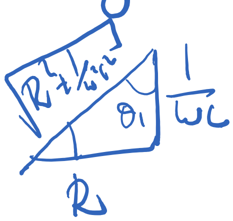
if $\quad i(t) = i_0 \cos(\omega t + \phi_i)$

$$V_C(t) = \frac{i_0}{\omega C} \cos(\omega t + \phi_i - \frac{\pi}{2})$$

$$V_R(t) = R i_0 \cos(\omega t + \phi_i)$$

$$V_0 \cos(\omega t) = V_R + V_C$$

$$V_0 \cos(\omega t) = i_0 \, R \cos(\underbrace{\omega t + \phi_i}) + \frac{i_0}{\omega C} \sin(\underbrace{\omega t + \phi_i})$$

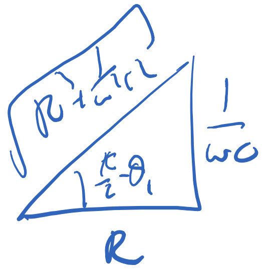



$$\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1 = \sin(\theta_1 + \theta_2)$$

$$= i_0 \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \left[ \underbrace{\sin\theta_1 \cos(\omega t + \phi_i) + \cos\theta_1 \sin(\omega t + \phi_i)} \right]$$

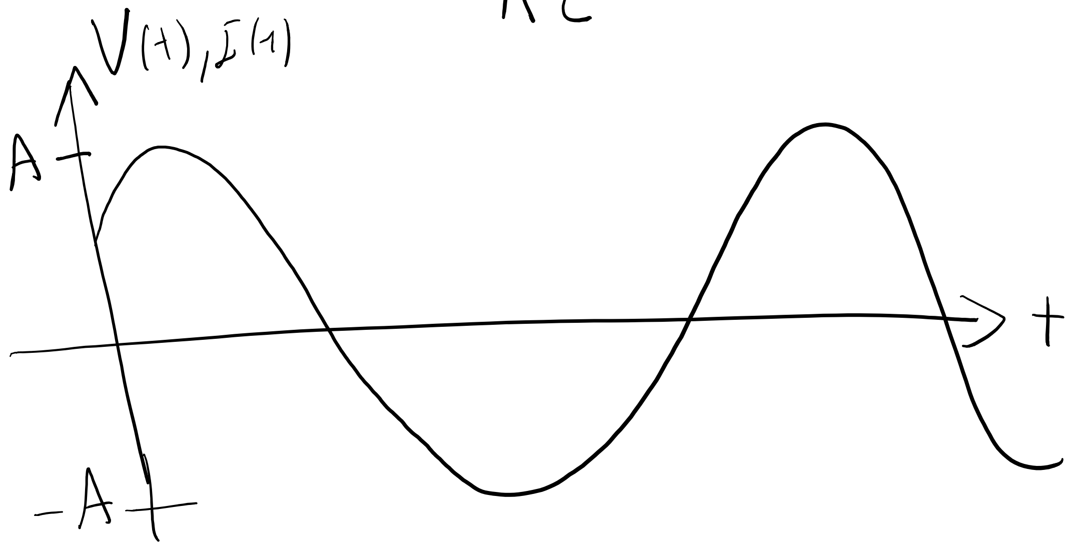
$$V_0 \cos(\omega t) = i_0 \sqrt{R^2 + \frac{1}{\omega^2 C^2}} \; \underbrace{\sin(\omega t + \phi_i - \theta_1)}_{\cos(\omega t + \phi_i - \theta_1 - \frac{\pi}{2})}$$

$$\boxed{i_0 = \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}}$$




$$\phi_i = \boxed{\arctan \frac{1}{\omega C R}}$$

AC




$$V(t) = A \cos(\omega t + \phi)$$

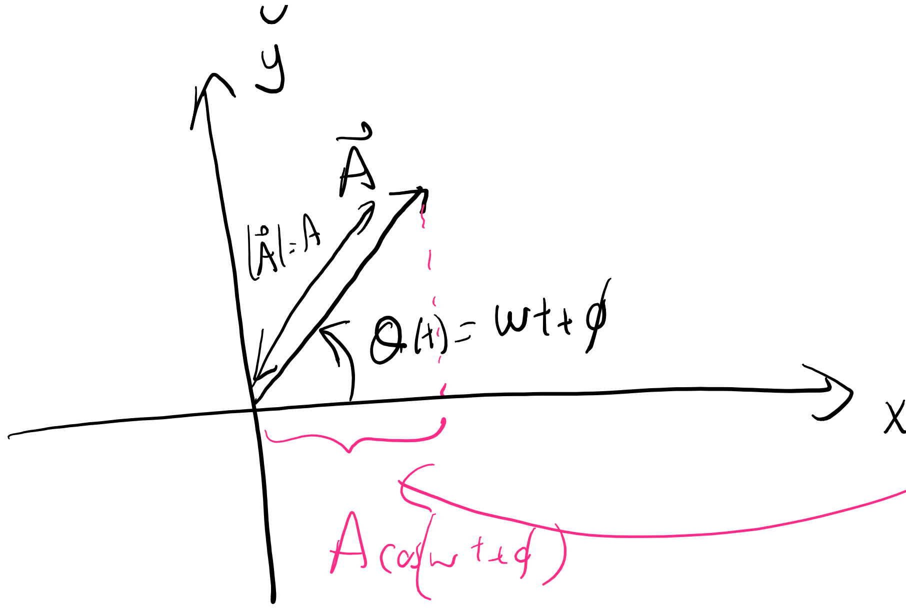
Amplitude

angular frequency

$T = \frac{2\pi}{\omega}$

phase

$\omega \rightarrow$ will be the same for all currents and voltages




$$A\cos(\omega t+\phi)$$

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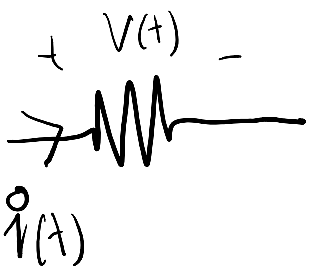
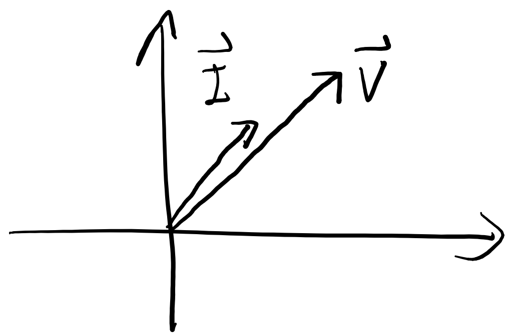
Resistor




$$V(t) = V_0 \cos(\omega t)$$ ~~Memristor~~

$$i(t) = \frac{V(t)}{R} = \frac{V_0}{R}\cos(\omega t)$$

$$Z = \frac{|\vec{V}|}{|\vec{I}|} = R$$

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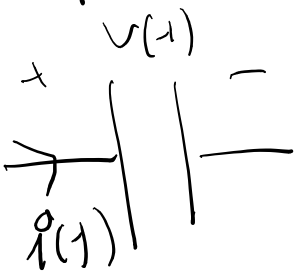
Capacitor

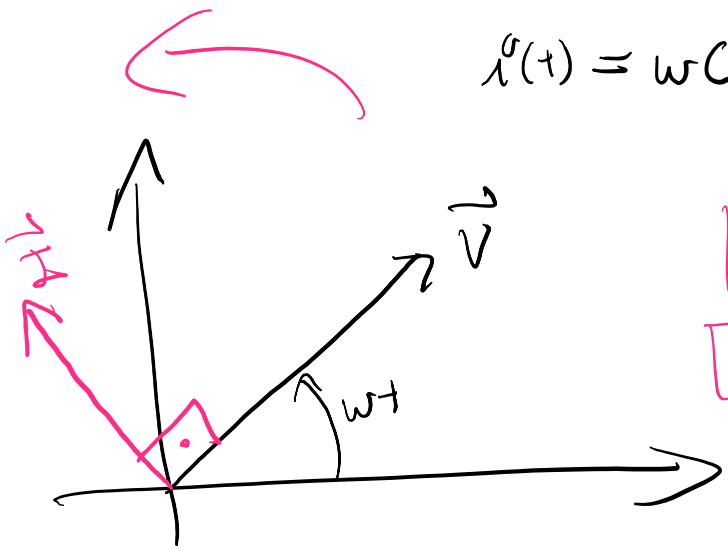



$$V(t) = V_0 \cos(\omega t)$$

$$i(t) = C\frac{dV(t)}{dt} = -CV_0\,\omega \sin(\omega t)$$

$$i(t) = -\omega C V_0 \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$i(t) = \omega C V_0 \cos\left(\omega t + \frac{\pi}{2}\right)$$




$$i(t) = \omega C V_0 \cos(\underbrace{\omega t + \tfrac{\pi}{2}})$$

$$\frac{|\vec{V}|}{|\vec{I}|} = \frac{V_0}{\omega C V_0} = \boxed{\frac{1}{\omega C} = Z}$$

I leads V

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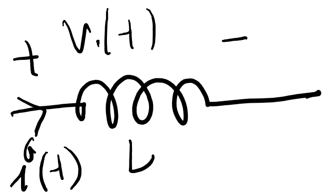
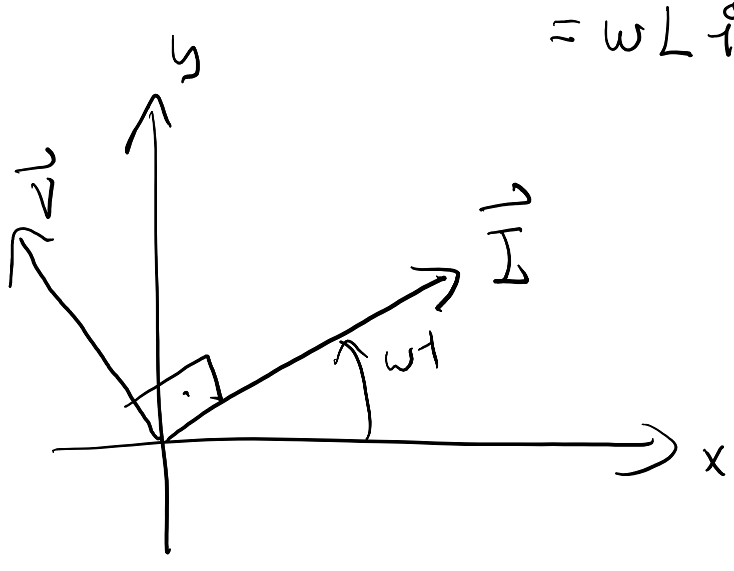



$$i(t) = i_0 \cos(\omega t)$$

$$V(t) = L\,\frac{di}{dt}$$

$$V(t) = -\omega L i_0 \sin(\omega t)$$

$$= \omega L i_0 \cos(\underbrace{\omega t + \tfrac{\pi}{2}})$$




V lead I

$$\frac{|\vec{V}|}{|\vec{I}|} = \frac{\omega L i_0}{i_0} = \boxed{Z = \omega L}$$

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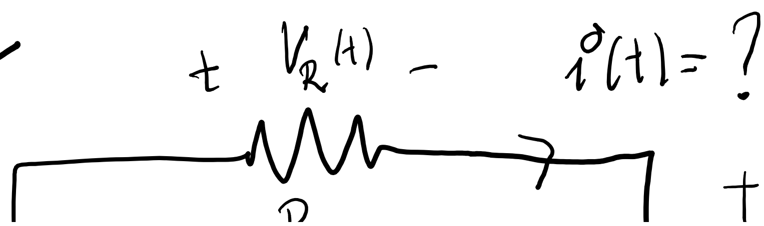
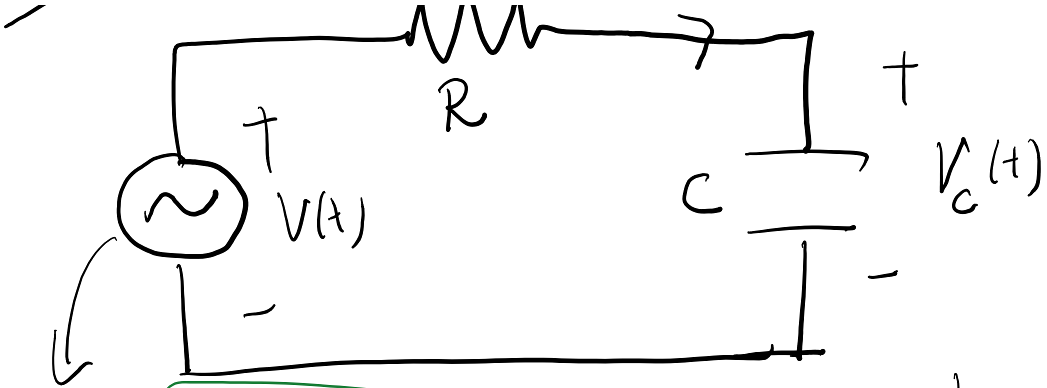
~~EX~~

$i(t) = ?$



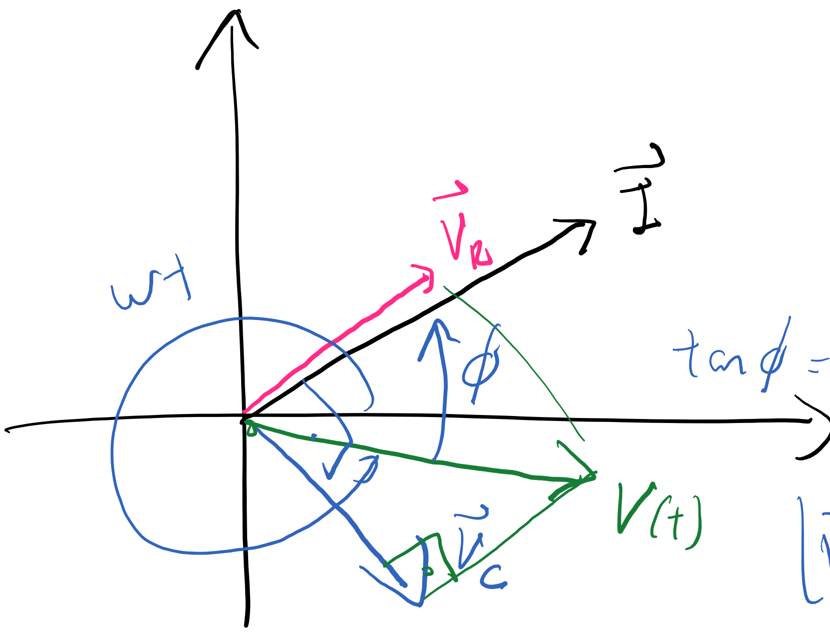



$V(t) = \boxed{V_0 \cos(\omega t)}$

· Find the average power dissipated in the resistor.

$V(t) = V_R(t) + V_C(t)$

$\vec{V}_R = R\vec{I}$




$|\vec{V}_0| = R\, i^0$

$\vec{V}_0$ ?

$\tan\phi = \frac{V_C}{V_R} = \frac{\frac{1}{\omega C} i^0}{R i^0} = \boxed{\frac{1}{\omega C R}}$

$|\vec{V}_0| = \frac{1}{\omega C} i^0$

$$\vec{V}(t) = \vec{V}_C + \vec{V}_R$$

$$|V(t)| = \sqrt{V_C^2 + V_R^2}$$

$$= \sqrt{R^2 i^{02} + \frac{1}{\omega^2 C^2} i^{02}} = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}\; i^0$$

$$\boxed{i_0 = \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}}$$

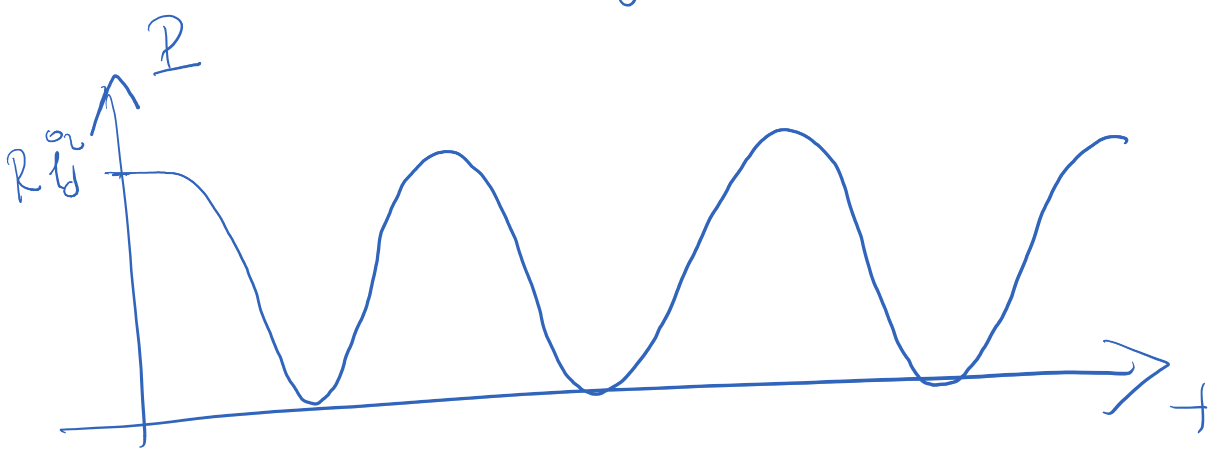
$$i(t) = \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \cos(\omega t + \phi)$$

$$\phi = \arctan\left(\frac{1}{\omega RC}\right)$$

$$P(t) = V_R(t)\, i(t)$$

$$= R\, i^2(t)$$

$$= R\, i_0^2 \cos^2(\omega t + \phi)$$




$$P_{avg} = \frac{1}{T} \int_0^T P(t)\, dt$$

$$1 \int^T (R i_0^2 \cos^2(\omega t + \phi))\, dt$$

$$= \frac{1}{T}\int_0^T R i_0^2 \cos^2(\omega t+\phi)\,dt$$

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$= \frac{R i_0^2}{T}\,\frac{1}{2}\int_0^T \left(\cos(2\omega t+2\phi)+1\right)dt$$

$$= \frac{R i_0^2}{2}\left[\underbrace{\frac{1}{T}\cancel{\left(\frac{1}{2\omega}\sin(2\omega t+2\phi)\right)\Big|_0^T}}_{0} + \underbrace{\frac{1}{T}\,t\Big|_0^T}_{1}\right]$$

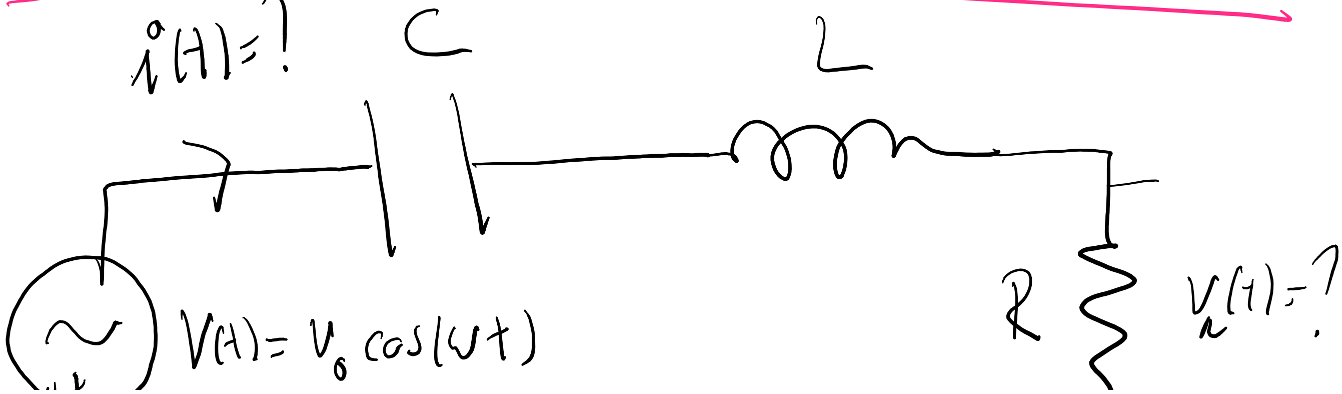
$$T = \frac{2\pi}{\omega} \qquad \frac{1}{4\pi}\left[\cancel{\underbrace{\sin\left(2\omega\frac{2\pi}{\omega}+\phi\right)}_{\sin\phi} - \sin(\phi)}\right]$$

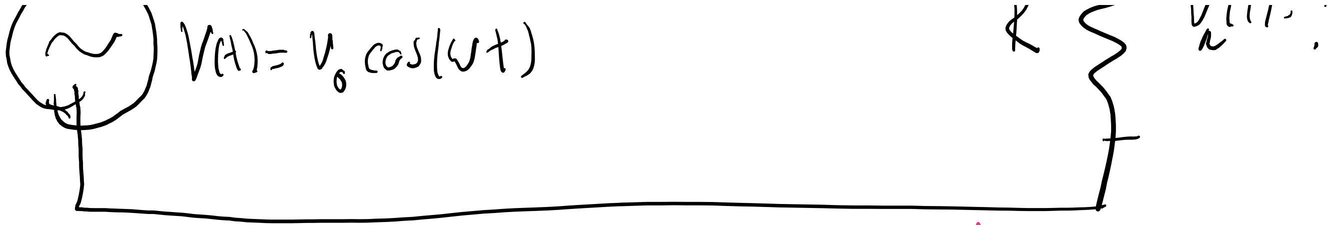
$$\boxed{P_{avg} = R\,\frac{i_0^2}{2}}$$

$$= R\, i_{RMS}^2$$

$$i_{RMS} = \frac{i_0}{\sqrt{2}}$$

root mean square

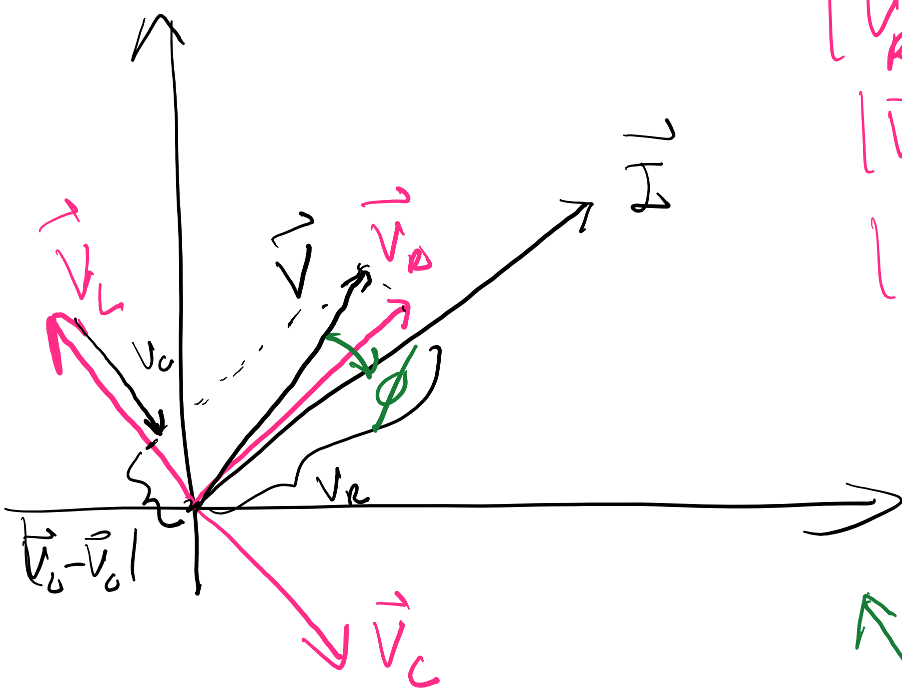


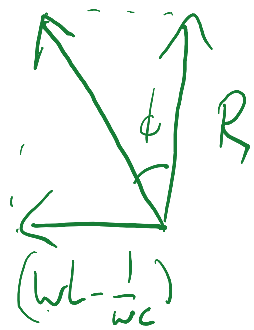



$$|\vec{V}_R| = R\, i_0$$

$$|\vec{V}_C| = \frac{1}{\omega C}\, i_0$$

$$|\vec{V}_L| = \omega L\, i_0$$




$$\vec{V} = \vec{V}_R + \vec{V}_C + \vec{V}_L$$

$$V_0 = |\vec{V}| = \sqrt{R^2 i_0^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 i_0^2}$$

$$\phi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

$$i_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$P = R\, \overline{i_0^2} = \frac{R\, V_0^2}{\phantom{x}}$$

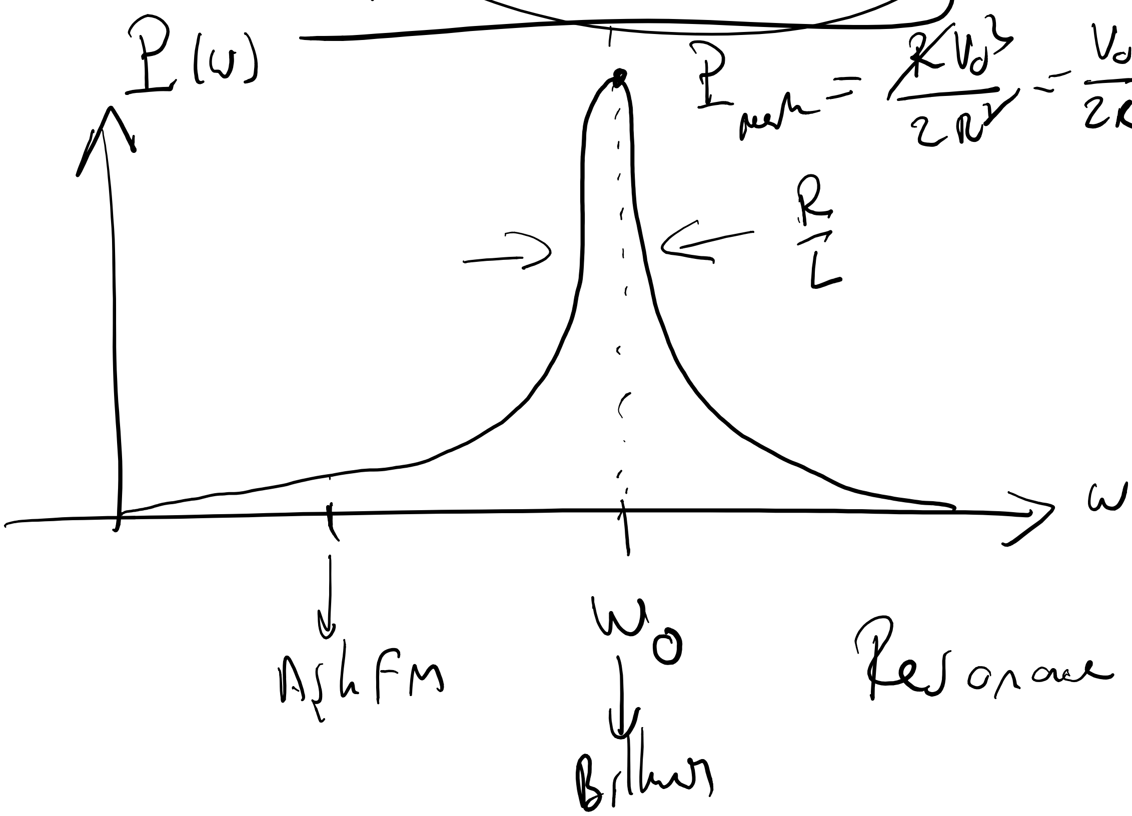
$$2 \qquad 2\left(R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right)$$

At which frequency is $P$ maximum?

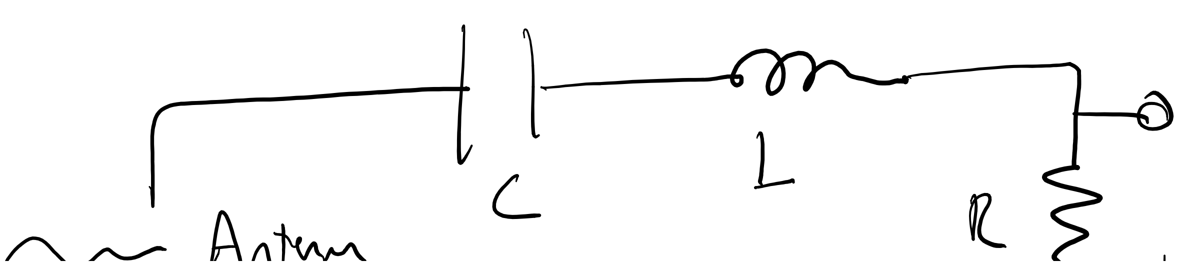
Clearly $\omega L = \frac{1}{\omega C}$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

max power




$$P_{peak} = \frac{\cancel{R} V_0^2}{2R^{\cancel{2}}} = \frac{V_0^2}{2R}$$




Antenna
C
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